On the weak topology in Lipschitz free spaces

Colin PETITJEAN

Lluís Santaló School 2023
Santander, Spain

Université Gustave Eiffel
LAMA LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES — UMR 8050

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Part 2

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Notation throughout this talk.

(M, d) is a **complete** metric space with a distinguished point $0 \in M$.

We let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and

$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{K} \text{ Lipschitz } : f(0) = 0\}.$$

When equipped with the norm

$$\|f\|_L = Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)},$$

it is a Banach space.

Then we consider the evaluation functional $\delta(x) : \mathrm{Lip}_0(M) \to \mathbb{K}$ defined by $\langle \delta(x), f \rangle = f(x)$, for every $f \in \mathrm{Lip}_0(M)$.

Definition

The *Lipschitz free space over M* is the following subspace of $\mathrm{Lip}_0(M)^*$:

$$\mathcal{F}(M) := \overline{\mathrm{span}}^{\|\cdot\|} \{\delta(x) \mid x \in M\}.$$

Some properties...

- $\delta : x \in M \mapsto \delta(x) \in \mathcal{F}(M)$ is an isometry.
- $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M) \rightsquigarrow$ weak topology $= \sigma(\mathcal{F}(M), \mathrm{Lip}_0(M))$.
- If $0 \in N \subset M$, then $\mathcal{F}(N) \equiv \mathcal{F}_M(N) := \overline{\mathrm{span}}\{\delta(x) \mid x \in N\} \subset \mathcal{F}(M)$.
- **"The intersection theorem" [Aliaga – Pernecká, 2020]:**

  If $(K_i)_{i\in I}$ is a family of closed subsets of M, then

  $$\bigcap_{i\in I} \mathcal{F}_M(K_i) = \mathcal{F}_M\Big(\bigcap_{i\in I} K_i\Big).$$

  $\rightarrow$ This leads to a notion of support for elements $\gamma \in \mathcal{F}(M)$:

  $S = \mathrm{supp}(\gamma)$ is the smallest closed subset of M such that $\gamma \in \mathcal{F}_M(S)$.
- [Godefroy – Kalton, 2003]: If X is a separable Banach space, then X is isometric to a subspace of $\mathcal{F}(X)$.

...and some classical examples.

- $M = T_1^\omega = \{0\} \cup \mathbb{N}$ the countably branching tree of height 1 (aka "the spider").

  Then $T\colon \delta(n) \in \mathcal{F}(M) \mapsto e_n \in \ell_1(\mathbb{N})$ is a surjective isometry.

- $(M, d) = (\mathbb{N}, |\cdot|)$. $T\colon \delta(n) \in \mathcal{F}(\mathbb{N}) \mapsto \sum_{i=1}^n e_i \in \ell_1(\mathbb{N})$ is a surjective isometry.

- $M = ([0,1], |\cdot|)$. $T\colon \delta(t) \in \mathcal{F}([0,1]) \mapsto \mathbb{1}_{[0,t]} \in L^1([0,1])$ is a surjective isometry.

- [Godard, 2010]: If $S \subset \mathbb{R}$, $\lambda(S) > 0$, then $L^1([0,1]) \hookrightarrow \mathcal{F}(S) \equiv L^1(\mu_S)$.

- [Naor – Schechtmann, 2007] (or [Kisljakov, 1975]): $\mathcal{F}(\mathbb{R}^2) \not\hookrightarrow L^1([0,1])$.

Question

$$\mathcal{F}(\mathbb{R}^2) \simeq \mathcal{F}(\mathbb{R}^3)?$$

A research program

- Try to characterize the (linear) properties of $\mathcal{F}(M)$ in terms of the (metric) properties of M.

In this talk, we are mainly interested in **properties** which are related to the **weak topology** of $\mathcal{F}(M)$.

$\rightarrow$ Schur property

$\rightarrow$ weak sequential completeness

$\rightarrow$ weak compactness

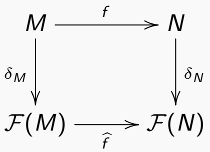
$\rightarrow$ ...

Research program bis

Try to characterize the (linear) properties of $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ in terms of the (metric) properties of $f : M \to N$.

Wait! What is $\widehat{f}$?

Proposition (Linearisation property)



$f : M \to N$ is Lipschitz s.t. $f(0_M) = 0_N$.

$$\widehat{f}\Big(\sum_{i=1}^{n} a_i \delta_M(x_i)\Big) = \sum_{i=1}^{n} a_i \delta_N\big(f(x_i)\big).$$

Examples:

- f is bi-Lipschitz if and only if $\widehat{f}$ is a linear embedding.
- f is a Lipschitz retraction if and only if $\widehat{f}$ is a linear projection.

Definition

$$\mathcal{FS}_k(M) := \{\gamma \in \mathcal{F}(M) \ : \ |\operatorname{supp}\gamma| \leq k\}.$$

Lemma (Aliaga – Noûs – Procházka – P., 2021)

The set $\mathcal{FS}_k(M)$ is weakly closed.

Theorem

If a sequence $(\gamma_n)_n \subset \mathcal{FS}_k(M)$ weakly converges to some $\gamma \in \mathcal{F}(M)$, then $\gamma \in \mathcal{FS}_k(M)$ and $(\gamma_n)_n$ converges to γ in the norm topology.

$\rightarrow$ A (rather elaborated) proof in the case $\gamma = 0$ can be found in a paper due to [Albiac – Kalton, 2009].
$\rightarrow$ For a more direct proof by induction, [Abbar – Coine – Petitjean, 2022].
$\rightarrow$ A (probably similar) proof is due to [Aliaga – Pernecká – Smith, ????]

Consequences: Let $f : M \to N$ be a Lipschitz map s.t. $f(0) = 0$.

Proposition (Abbar – Coine – P., 2022)

$$\widehat{f} \textit{ is compact} \iff \widehat{f} \textit{ is weakly compact.}$$

Proof.

- [Cabrera-Padilla – Jiménez-Vargas, 2016]: $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ is (weakly) compact if and only if

$$S := \left\{ \frac{\delta(f(x)) - \delta(f(y))}{d(x,y)} \mid x \neq y \in M \right\}$$

is relatively (weakly) compact in $\mathcal{F}(N)$.

- $S \subset \mathcal{FS}_2(M)$. □

Remark [Abbar – Coine – P., 2023]:
Also works for weighted versions $w\widehat{f}$ of these "Lipschitz operators".

$\to$ Leads to characterizations (in terms of metric properties of f) of those (weighted) Lipschitz operators $w\widehat{f}$ which are (weakly) compact.

Definition

A Banach space X has the ***Schur property*** if: $\forall (x_n)_n \subset X$,

$$x_n \xrightarrow{w} 0 \iff x_n \xrightarrow{\|\cdot\|} 0.$$

Examples:

- ℓ_1 has the Schur property (gliding hump technique).
- c_0 and ℓ_p $(p > 1)$ fail it since $e_n \xrightarrow{w} 0$ but $\|e_n\| = 1$, $\forall n \in \mathbb{N}$.
- X reflexive + Schur $\implies \dim(X) < \infty$.
- L^1 does not have it since it contains a copy of ℓ_2.

Theorem (Kalton, 2004)

If M is bounded and $0 < p < 1$ then $\mathcal{F}(M, d^p)$ has the Schur property.

Theorem (Kalton, 2004 – Upgraded version)

If $\mathrm{lip}_0(M)$ *separates the points of* M *uniformly, then* $\mathcal{F}(M)$ *has the Schur property.*

Lemma (Key argument)

If M *is bounded then and* $\gamma_n \xrightarrow{w} 0$ *then:* $\forall \varepsilon > 0$, $\forall \delta > 0$, $\exists E \subset M$ *finite such that:*

$$\sup_{n \in \mathbb{N}} \mathrm{dist}(\gamma_n, \mathcal{F}([E]_\delta)) < \varepsilon,$$

where $[E]_\delta = \{y \in M : d(y, E) \leq \delta\}$.

Remark:

$$\sup_{n \in \mathbb{N}} \mathrm{dist}(\gamma_n, \mathcal{F}([E]_\delta)) < \varepsilon \iff \{\gamma_n : n \in \mathbb{N}\} \subset \mathcal{F}([E]_\delta) + \varepsilon B_{\mathcal{F}(M)}.$$

Generalizing this proof, one can get:

Proposition (Aliaga – Noûs – Petitjean – Procházka, 2021)

Let W be a bounded set in $\mathcal{F}(M)$. Then:

$$W \text{ is weakly precompact } \implies W \text{ has Kalton's property.}$$

- A sequence $(x_n)_n$ is a Banach space X is ***weakly Cauchy*** if $(\langle x^*, x_n\rangle)_n$ is convergent for every $x^* \in X^*$.
- $W \subset X$ is ***weakly precompact*** whenever every sequence admits a weakly Cauchy subsequence.
- $W \subset \mathcal{F}(M)$ has ***Kalton's property*** if: $\forall \varepsilon > 0$, $\exists \delta > 0$, $\exists E \subset M$ finite s.t.

$$W \subset \mathcal{F}([E]_\delta) + \varepsilon B_{\mathcal{F}(M)}.$$
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- $W \subset \mathcal{F}(M)$ has Kalton's property if: $\forall \varepsilon > 0$, $\exists \delta > 0$, $\exists E \subset M$ finite s.t.

$$W \subset \mathcal{F}([E]_\delta) + \varepsilon B_{\mathcal{F}(M)}.$$

Theorem (Aliaga – Noûs – Procházka – P., 2021)

*If W has Kalton's property, then W is **tight**, that is: $\forall \varepsilon > 0$, $\exists K \subset M$ compact such that*

$$W \subset \mathcal{F}(K) + \varepsilon B_{\mathcal{F}(M)}.$$

Moreover, there exists a linear map $T : \operatorname{span} \delta(W) \to \mathcal{F}(K)$ s.t.

- $\|\mu - T\mu\| \leq \varepsilon$, $\forall \mu \in W$;
- *There is a sequence $(T_k)_k$ of bounded operators on $\mathcal{F}(M)$ such that $T_k \to T$ uniformly on W.*

Summary: Weak precompactness $\implies$ Kalton's property $\iff$ Tightness.

1. On the Schur property

Corollary (Aliaga – Noûs – Procházka – P., 2021)

$\mathcal{F}(M) \in (Schur) \iff \mathcal{F}(K) \in (Schur), \forall K \subset M$ *compact.*

If M is compact:

$$\mathcal{F}(M) \in (\mathrm{Schur}) \Longrightarrow L^1 \not\hookrightarrow \mathcal{F}(M) \overset{\text{[Godard]}}{\Longrightarrow} \left[S \subset \mathbb{R},\ \lambda(S) > 0 \implies S \underset{bi-Lip}{\not\hookrightarrow} M\right]$$

$$\left[S \subset \mathbb{R},\ \lambda(S) > 0 \implies S \underset{bi-Lip}{\not\hookrightarrow} M\right] \overset{\text{[Aliaga – Gartland – Procházka –P., 2022]}}{\Longrightarrow} \mathrm{lip}_0(M) \text{ S.P.U.} \overset{\text{[Kalton]}}{\Longrightarrow} \mathcal{F}(M) \in (\mathrm{Schur})$$

Therefore for general M:

$$\mathcal{F}(M) \in (\mathrm{Schur}) \iff L^1 \not\hookrightarrow \mathcal{F}(M) \iff \left[S \subset \mathbb{R},\ \lambda(S) > 0 \implies S \underset{bi-Lip}{\not\hookrightarrow} M\right]$$

2. On weak sequential completeness

Definition

A Banach space X is ***weakly sequentially complete*** if every weakly Cauchy sequence in X is actually weakly convergent.

Examples:

- (Schur) $\implies$ (w.s.c.)
- (Reflexivity) $\implies$ (w.s.c.)
- L^1 is w.s.c. (Dunford – Pettis theorem)
- c_0 is not w.s.c. (e.g. considering the summing basis)

Remark: Thanks to [Godefroy – Kalton, 2003], c_0 is isometric to a subspace of $\mathcal{F}(c_0)$, and therefore $\mathcal{F}(c_0)$ is not w.s.c.

Corollary (Aliaga – Noûs – Procházka – P., 2021)

$\mathcal{F}(M)$ is w.s.c. $\iff$ $\mathcal{F}(K)$ is w.s.c., $\forall K \subset M$ compact.

- **[Cuth – Doucha – Wojtaszczyk, 2016]**: $\mathcal{F}([0,1]^n) \hookrightarrow C^1([0,1]^n)^*$ and so it is w.s.c. by a result of **[Bourgain, 1983]**.
- **[Kochanek – Pernecká, 2018]**: If K is a compact subset of a superreflexive space S, then $\mathcal{F}(K)$ is w.s.c.

Consequences:

- $\mathcal{F}(S)$ is w.s.c. for every superreflexive space S.
- For every $p \in (1, \infty)$, $\mathcal{F}(\ell_p) \not\simeq \mathcal{F}(c_0)$.

Questions 1: Find a characterization (in terms of properties of M) of those Lipschitz free spaces $\mathcal{F}(M)$ which have:

- A quantitative version of the Schur property;
- The Dunford–Pettis property.
- ...

Remark: The Dunford-Pettis property is also *"compactly determined"* in free spaces, but careful with the statement: $\mathcal{F}(M)$ has the (DPP) if and only if for every compact $K \subset M$ there is a subset $B \subset M$ such that $K \subset B$ and $\mathcal{F}(B)$ has the (DPP).

Question 2: Is $\mathcal{F}(\ell_1)$ w.s.c.? Is it true that $c_0 \hookrightarrow \mathcal{F}(\ell_1)$?

Question 3: Find a characterization of weakly (pre)compact subsets of $\mathcal{F}(M)$.

Recall that for a bounded $W \subset \mathcal{F}(M)$:

$$W \text{ is weakly precompact} \implies W \text{ is tight} \iff W \text{ has Kalton's property.}$$

But the converse is trivially false: if M is compact then any $W \subset \mathcal{F}(M)$ (e.g. $W = B_{\mathcal{F}(M)}$) is tight!

To be continued...

Muchas gracias por su atención!