Some remarks about “Lipschitz-free operators”

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Ongoing work...

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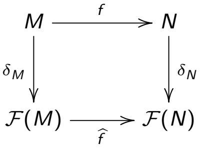
Let M and N be two *pointed* metric spaces with basepoints $0_M \in M$ and $0_N \in N$.

Let $f\colon M \to N$ be a Lipschitz map such that $f(0_M) = 0_N$.

Then there exists:

1. two (unique) Banach spaces $\mathcal{F}(M)$ and $\mathcal{F}(N)$ together with isometries $\delta_M : M \to \mathcal{F}(M)$ and $\delta_N : N \to \mathcal{F}(N)$ (ranges are linearly dense)
2. a linear bounded operator $\widehat{f}\colon \mathcal{F}(M) \to \mathcal{F}(N)$ with $\|\widehat{f}\| = Lip(f)$,

such that the following diagram commutes:



That is $\widehat{f} \circ \delta_M = \delta_N \circ f$.

Terminology: We call $\widehat{f}$ a *Lipschitz-free operator* or simply a *Lipschitz operator*.

“Program”:

Characterise the (linear) properties of $\widehat{f}$ in terms of the (metric) properties of f.

In this talk, we will talk about

- some dynamical properties (transitivity, hypercyclicity, etc.)
- some compactness properties
- injectivity

May be compared with a “more classical” research program in Lipschitz-free spaces theory:

Characterise the (linear) properties of $\mathcal{F}(M)$ in terms of the (metric) properties of M.

1. A short introduction to Lipschitz-free spaces

2. Dynamical properties

3. Compactness

4. Injectivity

One way, among others, to define the Lipschitz free spaces over M.

Let (M, d) be a metric space with a distinguished point $0 \in M$.

Let X be a (real) Banach space.

We let

$$\mathrm{Lip}_0(M, X) = \{f : M \to X \text{ Lipschitz} \mid f(0) = 0\}$$

When equipped with the norm

$$\|f\|_L = Lip(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|_X}{d(x, y)},$$

it is a Banach space.

Notation: $\mathrm{Lip}_0(M) := \mathrm{Lip}_0(M, \mathbb{R})$

Then we consider the evaluation functional $\delta(x) : \mathrm{Lip}_0(M) \to \mathbb{R}$ defined by $\langle \delta(x), f \rangle = f(x)$, for every $f \in \mathrm{Lip}_0(M)$.

Definition

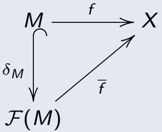
The Lipschitz-free space over M is the following subspace of $\mathrm{Lip}_0(M)^*$:

$$\mathcal{F}(M) := \overline{\mathrm{span}}^{\|\cdot\|} \{\delta(x) \mid x \in M\}.$$

Proposition (Fundamental extension property)

For every Banach space X, for every $f \in \mathrm{Lip}_0(M, X)$, the unique linear operator $\overline{f} \colon \mathcal{F}(M) \to X$ defined on $\operatorname{span} \delta(M)$ by

$$\overline{f}\Big(\sum_{i=1}^{n} a_i \delta(x_i)\Big) = \sum_{i=1}^{n} a_i f(x_i) \in X$$



is continuous with $\|\overline{f}\| = Lip(f)$.

Consequences:

- $\mathrm{Lip}_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X)$
- $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$

Remarks:

1. If $0 \in N \subset M$, then $\mathcal{F}(N) = \overline{\operatorname{span}}\{\delta(x) \mid x \in N\} \subset \mathcal{F}(M)$.
2. M will always be complete $(\mathcal{F}(M) \equiv \mathcal{F}(\overline{M}))$.
3. A change of the base point in M does not affect the isometric structure of $\mathcal{F}(M)$.
4. There is a notion of support for elements $\gamma \in \mathcal{F}(M)$: $S = \operatorname{supp}(\gamma) \subset M \iff S$ is the smallest closed subset of M such that $\gamma \in \mathcal{F}(S)$.

... and some important features.

Corollary (Linearisation property)

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \delta_N} \\ \mathcal{F}(M) & \xrightarrow[\ \widehat{f}\ ]{} & \mathcal{F}(N) \end{array}$$

$$(f(0_M) = 0_N)$$

$$\widehat{f}\Big(\sum_{i=1}^{n} a_i \delta_M(x_i)\Big) = \sum_{i=1}^{n} a_i \delta_N(f(x_i))$$

Remark:

The dual operator of $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ can be naturally identified with a composition (by f) operator between the Lipschitz spaces $\mathrm{Lip}_0(N)$ and $\mathrm{Lip}_0(M)$.

Indeed, if we let $C_f : g \in \mathrm{Lip}_0(N) \mapsto g \circ f \in \mathrm{Lip}_0(M)$ then one has:

$$\langle (\widehat{f}\,)^*(g), \delta(x) \rangle = \langle g, \widehat{f}(\delta(x)) \rangle = \langle g, \delta(f(x)) \rangle = g \circ f(x) = \langle C_f(g), \delta(x) \rangle.$$

Examples:

1. $(M, d) = (\mathbb{N}, |\cdot|)$. $T : \delta(n) \in \mathcal{F}(\mathbb{N}) \mapsto \sum_{i=1}^{n} e_i \in \ell_1(\mathbb{N})$ is a surjective isometry.
2. $M = ([0,1], |\cdot|)$. $T : \delta(t) \in \mathcal{F}([0,1]) \mapsto \mathbb{1}_{[0,t]} \in L^1([0,1])$ is a surjective isometry.

How the properties of f and $\hat{f}$ are related?

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & N \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ \mathcal{F}(M) & \xrightarrow[\hat{f}]{} & \mathcal{F}(N) \end{array}$$

- f is bi-Lipschitz if and only if $\hat{f}$ is a linear into isomorphism (i.e. linear embedding).
- f is a Lipschitz isomorphism (bi-Lipschitz and surjective) if and only if $\hat{f}$ is a linear isomorphism.
- f has dense range if and only if $\hat{f}$ has dense range.
- f is a Lipschitz retraction if and only if $\hat{f}$ is a linear projection.

Let $f : M \to M$ and $x \in M$. *The orbit of* x *under* f is defined by

$$\mathrm{Orb}(x, f) := \{f^n x : \ n \in \mathbb{N} \cup \{0\}\}.$$

Definition

We will say that:

1. f is *hypercyclic* if there exists $x \in M$ such that $\overline{\mathrm{Orb}(x, f)} = M$.
2. f is *topologically transitive* if, for each pair of nonempty open sets U, V of M, there exists $n \in \mathbb{N} \cup \{0\}$ such that $f^n(U) \cap V \neq \emptyset$.

- If M has no isolated point then any hypercyclic map is topologically transitive.

  Proof. $\overline{\mathrm{Orb}(x, f)} = M \implies \exists m \geq 0, \ f^m(x) \in U$.
  $\mathrm{Orb}(x, f) \setminus \{x, f(x), \dots, f^m(x)\}$ is still dense in M.
  $\implies \exists n \geq 0$ such that $f^n(f^m(x)) = f^{n+m}(x) \in V$.
  $\implies f^n(U) \cap V \neq \emptyset$

- Conversely, if M is a separable complete space then a topologically transitive map is hypercyclic (Birkhoff transitivity theorem).

  A classical proof uses the Baire category theorem to prove that the set of points in M which have dense orbit is dense G_δ-set.

We will also consider the next definitions for a **linear** D.S. (X, T):

Definition

A bounded operator $T : X \to X$ is *cyclic* if there exists a vector $x \in X$ such that $\operatorname{span} \mathrm{Orb}(x, f)$ is dense in X.

Clearly:

$$\text{Hypercyclicity} \Rightarrow \text{Cyclicity.}$$

Remark: These notions are linked to the invariant subspace problem:
"Does every bounded operator T on X admits a non-trivial invariant closed subspace?" (Open in the reflexive case)

Notice that:

- T does not have any invariant closed subspace $\iff$ every $x \in X \setminus \{0\}$ is a cyclic vector.
- T does not have any invariant closed subset $\iff$ every $x \in X \setminus \{0\}$ is a hypercyclic vector.

Some observations from [1a]:

Lemma

1. *For every* $n \in \mathbb{N}$, $\widehat{f^n} = (\widehat{f})^n$.
2. *For every* $x \in M$, $\mathrm{Orb}(\delta(x), \widehat{f}) = \delta(\mathrm{Orb}(x, f))$.

- x is a hypercyclic element for $f \iff \delta(x)$ is a cyclic vector for $\widehat{f}$.
- If γ is a hypercyclic vector for $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(M)$, then γ must be infinitely supported (i.e. $\gamma \notin \operatorname{span} \delta(M)$).
- If $\mathrm{Per}(f)$ of f is dense in M, then $\mathrm{Per}(\widehat{f})$ is dense in $\mathcal{F}(M)$.

  We recall that x is a periodic point of f if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$. (related to Chaos in the sense of Devaney).
- If $f : [a, b] \to [a, b]$ is Lipschitz and topologically transitive (i.e. hypercyclic), with a fixed point $c \in [a, b]$, then $\widehat{f}$ is hypercyclic.

  In fact, $\widehat{f}$ is weakly mixing (and it might even be mixing).

A map $f : M \to M$ is said to be

- *(topologically) weakly mixing* if $f \times f$ is topologically transitive on $M \times M$, that is, for every nonempty open sets U_1, U_2, V_1, V_2 of M, there exists $n \in \mathbb{N} \cup \{0\}$ such that $f^n(U_1) \cap V_1 \neq \emptyset$ and $f^n(U_2) \cap V_2 \neq \emptyset$;
- *(topologically) mixing* if for each pair of nonempty open sets U, V of M there exists $N \in \mathbb{N} \cup \{0\}$ such that for every $n \geq N$, $f^n(U) \cap V \neq \emptyset$.

Back on M pointed metric space and $f : M \to M$ is Lipschitz with $f(0) = 0$.

[M. Murillo-Arcila and A. Peris, (2015)]: As a consequence of a more general theorem, they obtain:

$$f \text{ mixing / weakly mixing} \implies \widehat{f} \textit{mixing} \text{ / weakly mixing,}$$

Remark : Both reverse implications are false : Even on $[a, b]$, there exists f non transitive such that $\widehat{f}$ is mixing.
(but f^2 is transitive...)

What else?

Question

Is it possible to build a Lipschitz operator with no non-trivial invariant subspace?

1. A short introduction to Lipschitz-free spaces
2. Dynamical properties
3. Compactness
4. Injectivity

A bounded operator $T : X \to Y$ between Banach spaces is (weakly) compact if $T(B_X)$, is relatively (weakly) compact in Y.

Theorem (Jiménez-Vargas and Villegas-Vallecillos; 2013)

Let M be bounded and separable.
Let $f : M \to M$ be a Lipschitz map vanishing at 0_M.
Then the composition operator $C_f : g \in \mathrm{Lip}_0(M) \mapsto g \circ f \in \mathrm{Lip}_0(M)$ is compact if and only if

(i) *$f(M)$ is totally bounded in M.*
(ii) *f is uniformly locally flat, that is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(y)) \leq \varepsilon d(x, y)$ whenever $d(x, y) \leq \delta$.*

Remarks:

- By Schauder's theorem, the same characterization holds for $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(M)$.
- The very same result holds for Lipschitz maps $f : M \to N$.
- The proof does not use Lipschitz-free spaces at all...

[A. Jiménez-Vargas, J. M. Sepulcre and M. Villegas-Vallecillos; 2014]: The case when $N = Y$ is a Banach space is considered, and a characterisation is given in terms of "$\widehat{f}(\textit{Molecules})$ is relatively compact".

Proposition (M. G. Cabrera-Padilla, and A. Jiménez-Vargas; 2016)

Let M, N be pointed metric spaces and let $f : M \to N$ be a base point-preserving Lipschitz mapping. Then $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ is (weakly) compact if and only if

$$\left\{ \frac{\delta(f(x)) - \delta(f(y))}{d(x,y)} \mid x \neq y \in M \right\}$$

is relatively (weakly) compact in $\mathcal{F}(N)$.

Proof. " $\implies$ " Let $\mathcal{M} = \left\{ d(x,y)^{-1}(\delta(x) - \delta(y)) \mid x \neq y \in M \right\}$. Notice that

$$\left\{ \frac{\delta(f(x)) - \delta(f(y))}{d(x,y)} \mid x \neq y \in M \right\} = \widehat{f}(\mathcal{M}),$$

Since $\mathcal{M} \subset B_{\mathcal{F}(M)}$, if $\widehat{f}$ is compact then $\widehat{f}(\mathcal{M})$ must be relatively compact.
" $\impliedby$ " Hahn–Banach separation theorem: $B_{\mathcal{F}(M)} = \overline{\text{conv}}\mathcal{M}$
Now observe that

$$\widehat{f}(B_{\mathcal{F}(M)}) \subset \widehat{f}(\overline{\text{conv}}\mathcal{M}) \subset \overline{\text{conv}}(\widehat{f}(\mathcal{M})) \subset \overline{\text{conv}}\left(\overline{\widehat{f}(\mathcal{M})}\right).$$

So, if $\widehat{f}(\mathcal{M})$ is relatively compact, then $\overline{\text{conv}}\left(\overline{\widehat{f}(\mathcal{M})}\right)$ is compact and therefore $\widehat{f}(B_{\mathcal{F}(M)})$ is relatively compact.

Theorem ([1b])

Let M, N be any pointed metric spaces.
Let $f \in \mathrm{Lip}_0(M, N)$.
Then $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ is compact if and only if the next assertions are satisfied:

(P_1) *For every bounded subset $S \subset M$, $f(S)$ is totally bounded in N;*

(P_2) *f is uniformly locally flat, that is,*

$$\lim_{d(x,y)\to 0} \frac{d(f(x), f(y))}{d(x,y)} = 0;$$

(P_3) *For every $(x_n, y_n)_n \subset \widetilde{M} := \{(x,y) \in M \times M \mid x \neq y\}$ such that $\lim\limits_{n\to\infty} d(x_n, 0) = \lim\limits_{n\to\infty} d(y_n, 0) = \infty$, either*

- *$(f(x_n), f(y_n))_n$ has an accumulation point in $N \times N$, or*
- $\liminf\limits_{n\to+\infty} \dfrac{d(f(x_n), f(y_n))}{d(x_n, y_n)} = 0.$

Proof: Quite elementary, once we have some structural results about sequences $(\gamma_n)_n \subset \mathcal{F}(M)$ such that $|\operatorname{supp} \gamma_n| \leq 2$.

In fact, most of the time we do not use norm convergence but rather weak convergence of these kind of sequences...

Theorem

Let M be a complete metric space. Let $(\gamma_n)_n \subset \mathcal{F}(M)$ be a sequence such that $k := \sup_n |\operatorname{supp} \gamma_n| < \infty$. If $(\gamma_n)_n \subset \mathcal{F}(M)$ weakly converges to some $\gamma \in \mathcal{F}(M)$, then $|\operatorname{supp} \gamma| \leq k$ and $(\gamma_n)_n$ actually converges to γ in the norm topology.

Proof: Enough to mix a **deep result** by Albiac and Kalton (2009), and the fact that $\{\gamma \in \mathcal{F}(M) \ : \ |\operatorname{supp} \gamma| \leq k\}$ is weakly closed ([1a]).

[1b]: New more elementary proof by induction on k.

Theorem ([1b])

Let M, N be complete pointed metric spaces, and let $f : M \to N$ be a base point-preserving Lipschitz mapping. The the next conditions are equivalent

1. $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ *is compact;*
2. $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ *is weakly compact;*

Proof: $\widehat{f}$ compact $\iff \widehat{f}(\mathcal{M})$ rel. compact

$\iff \widehat{f}(\mathcal{M})$ rel. weakly seq. compact

$\iff \widehat{f}(\mathcal{M})$ rel. weakly compact *(Eberlein–Šmulian theorem)*

$\iff \widehat{f}$ weakly compact

Theorem ([1b])

Let M, N be complete pointed metric spaces, and let $f : M \to N$ be a base point-preserving Lipschitz mapping. The the next conditions are equivalent

1. $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ *is compact;*
2. $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ *is weakly compact;*
3. $C_f : \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$ *is compact;*
4. $C_f : \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$ *is weakly compact;*

Proof: (1) $\iff$ (3) follows from Schauder's theorem
(2) $\iff$ (4) follows from Gantmacher's theorem

Remark: This generalizes a result due to A. Jiménez-Vargas (2015) who proved (3) $\iff$ (4) when M is a compact metric space such that $\mathrm{lip}_0(M)$ is a norming subspace of $\mathrm{Lip}_0(M)$ (for $\mathcal{F}(M)$), where $\mathrm{lip}_0(M)$ is the subspace of all uniformly locally flat Lipschitz functions $M \to \mathbb{R}$.

[Aliaga-Gartland-Petitjean-Procházka, 2021]: For compact M

$$\mathrm{lip}_0(M) \text{ is norming } \iff \mathcal{F}(M) \equiv \mathrm{lip}_0(M)^* \iff M \text{ is purely 1-unrectifiable,}$$

where M p1u means that it contains no *curve fragment* ($\gamma\colon K \to M$ bi-Lipschitz embedding with $K \subset \mathbb{R}$ compact with $\lambda(K) > 0$).

One implication is clear: Assume that f is not injective.
There exists $x \neq y$ such that $f(x) = f(y)$.
This implies that: $\langle \widehat{f}, \delta(x) \rangle = \langle \widehat{f}, \delta(y) \rangle$,
showing that $\widehat{f}$ is not injective.

Therefore, $\widehat{f}$ injective $\implies$ f injective, and it remains one implication to study:

Question

f injective $\implies$ $\widehat{f}$ injective?

Some answers ([2]):

- Not true in general, e.g., there exists $f : [0,1] \to [0,1]$ which is injective but $\widehat{f}$ is not injective.
- There are some sufficient conditions on f which imply that $\widehat{f}$ is injective: f biLipschitz, f locally bi-Lipschitz + a non returning condition at every x ($\exists r, \rho > 0$ such that $f|_{B(x,r)}$ is bi-Lipschitz and $f^{-1}(B(f(x), \rho)) \subset B(x, r)$), and some others...
- For some metric spaces M, every Lipschitz map $f : M \to N$ (for any N) admits an injective linearization. We will say that M is *Lip-lin injective*.

There exists $f : [0,1] \to [0,1]$ which is injective but $\widehat{f} : \mathcal{F}([0,1]) \to \mathcal{F}([0,1])$ is not injective.

Remember that $T : \delta(t) \in \mathcal{F}([0,1]) \mapsto \mathbb{1}_{[0,t]} \in L^1([0,1])$ is a surjective isometry.

$$\begin{array}{ccc} \mathcal{F}([0,1]) & \xrightarrow{\widehat{f}} & \mathcal{F}([0,1]) \\ \big\downarrow T & & \big\downarrow T \\ L^1([0,1]) & \xrightarrow[\Phi_f]{} & L^1([0,1]) \end{array}$$

For every $\varphi \in L_1([0,1])$ we have

$$\Phi_f(\varphi) = \varphi \circ f^{-1}.$$

Let $C \subset [0,1]$ be closed, totally disconnected such that $\lambda(C) \in (0,1)$, $\min C = 0$ and $\max C = 1$ (e.g. "fat Cantor set").
We define $f : ([0,1], |\cdot|) \to ([0,1], |\cdot|)$ as

$$f(x) = \lambda([0,x] \setminus C) = \int_0^x 1_{[0,1]\setminus C}(t)dt.$$

Then f is 1-Lipschitz, non-decreasing, $f(0) = 0$ and $f(1) = 1 - \lambda(C) > 0$.
Moreover f is injective: If $x < y$, there exist $a < b$ in (x,y) such that $[a,b] \cap C = \emptyset$.
Thus $f(y) - f(x) = \lambda([x,y] \setminus C) \geq b - a > 0$. So f is injective.
Finally, a simple integration by substitution gives

$$\lambda(f(C)) = \int_{f(C)} 1dt = \int_C f'(x)dx = \int_C 1_{[0,1]\setminus C}(x)dx = 0.$$

Therefore $0 \neq 1_C \in L_1[0,1]$ but $\Phi_f(1_C) = 1_C \circ f^{-1} = 1_{f(C)} = 0 \in L_1[0,1]$. $\square$

Why is the example not simpler? (f being injective)

- Clear: If $\gamma \in \ker(\widehat{f})$ then $\gamma \notin \operatorname{span} \delta(M)$.
- But also $\gamma \neq \sum_{n=1}^{\infty} a_n \delta(x_n)$ where $(a_n) \in \ell_1$ and (x_n) pairwise-different.

  (In our counterexample γ can be expressed as $\gamma = \delta(1) - \sum_n \delta(x_n) - \delta(y_n)$ where $\lim_{n\to\infty} y_n - x_n = 0$ fast enough).
- The choice of f cannot be much simpler because of the sufficient conditions implying that $\widehat{f}$ is injective.
- The choice of M cannot be much simpler, e.g., if M is uniformly discrete or if M is compact with $\mathcal{H}^1(M) = 0$ then M is Lip-lin injective.

Remarks: The above construction can be "adapted" in order to show that:

1. If $M \subset \mathbb{R}$ such that $\lambda(M) > 0$, then M is not Lip-lin injective;
2. If M be a metric space which is not p1u, then M is not Lip-lin injective;
3. Being compact, p1u and totally disconnected is not sufficient to be Lip-lin injective;
4. There exists a countable, discrete, complete M which is not Lip-lin injective.

To conclude, an interesting property of an injective $\widehat{f}$: *"it preserves supports"*.

Proposition ([2])

Let $f \in \mathrm{Lip}_0(M, N)$*. Then, for any* $\gamma \in \mathcal{F}(M)$*,*

$$\mathrm{supp}\left(\widehat{f}(\gamma)\right) \subset \overline{f\left(\mathrm{supp}(\gamma)\right)}.$$

The inclusion is strict whenever $\widehat{f}$ is non-injective: if $\gamma \neq 0 \in \mathcal{F}(M)$ is such that $\widehat{f}(\gamma) = 0$, then $\mathrm{supp}\, \widehat{f}(\gamma) = \mathrm{supp}\, 0 = \emptyset$ while $f(\mathrm{supp}(\gamma)) \neq \emptyset$.

Theorem ([2])

If M *is bounded and* $f \in \mathrm{Lip}_0(M, N)$ *then* $\hat{f}$ *is injective if and only if* f *preserves supports, that is,* $\mathrm{supp}(\hat{f}(\gamma)) = \overline{f(\mathrm{supp}\, \gamma)}$.

[1*a*]: On the dynamics of Lipschitz operators,
with Arafat Abbar and Clément Coine, Integral Equations Operator Theory 93 (2021), no. 4, Paper No. 45, 27 pp.

[1*b*]: Compact and weakly compact Lipschitz operators,
with Arafat Abbar and Clément Coine, preprint (2021), arXiv:2110.03231.

[2]: Lipschitz operators which preserves injectivity,
with Luis García-Lirola and Antonín Procházka, to appear (soon?) on arXiv.

Thank you for your attention!