Some remarks on the dynamics of Lipschitz operators

Colin PETITJEAN

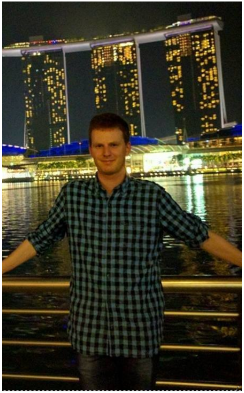
Journées du GDR AFHP 2020

Université Gustave Eiffel

LAMA LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES — UMR 8050

Ongoing work, joint with...

Arafat Abbar



Clément Coine

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"Program":

Characterise the (linear) properties of $\widehat{f}$ in (metric) terms of the properties of f.

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3. One advantage: Some definitions make sense for non-linear maps.

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  A classical proof uses the Baire category theorem to prove that the set of points in M which have dense orbit is dense G_δ-set.

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Conversely, if $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(M)$ has a given dynamical property, what can be said about $f : M \to M$?

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For every Banach space X, for every $f \in \mathrm{Lip}_0(M,X)$, the unique linear operator $\overline{f} \colon \mathcal{F}(M) \to X$ defined on $\operatorname{span} \delta(M)$ by

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Corollary (Linearisation property)

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\delta_M"'] & N \arrow[d, "\delta_N"] \\ \mathcal{F}(M) \arrow[r, "\widehat{f}"'] & \mathcal{F}(N) \end{tikzcd}$$

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1. $(M, d) = (\mathbb{N}, |\cdot|)$. The linear operator satisfying

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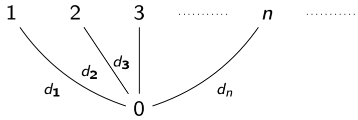

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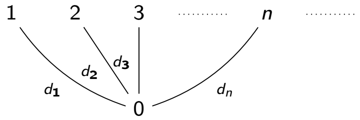

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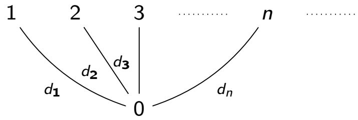

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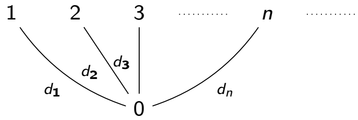

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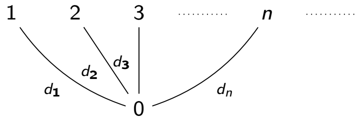

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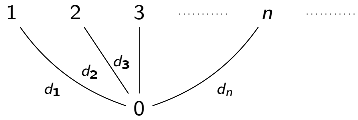

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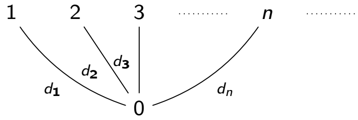

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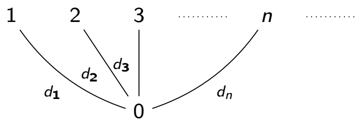

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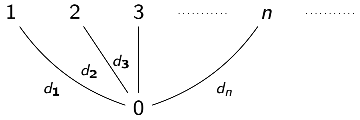

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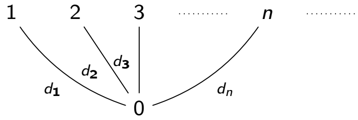

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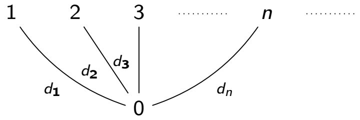

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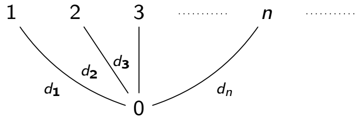

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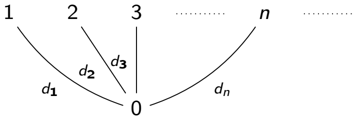

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Thus span $\mathrm{Orb}(\gamma, \widehat{f})$ cannot be dense in $\mathcal{F}(M)$.

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Proof. Consequence of the fact that $FS_n(M) := \{\gamma \in \mathcal{F}(M) \ : \ |\operatorname{supp}\gamma| \leq n\}$ is closed. □

Definition

- f is *(topologically) mixing* if for each pair of nonempty open sets U, V of M there exists $N \in \mathbb{N} \cup \{0\}$ such that for every $n \geq N$, $f^n(U) \cap V \neq \emptyset$;

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Then $\widehat{f}$ satisfies the hypercyclicity criterion. In particular, $\widehat{f}$ is hypercyclic.

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But for every $(n_k)_k$, $\liminf\limits_{k \to +\infty} d(f^{n_k}(\frac{1}{n}), 0) \geq \frac{1}{2}$ and so f fails the HCL. $\square$

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Corollary

Let $f : [a, b] \to [a, b]$ be a Lipschitz and topologically transitive map with a fixed point $c \in [a, b]$. Then $\hat{f}$ is Devaney chaotic.

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If $f : [a, b] \to [a, b]$ is a Lipschitz and topologically transitive map with a fixed point $c \in [a, b]$, then $\widehat{f}$ is weakly mixing.
If moreover f admits at least two fixed points, then $\widehat{f}$ is mixing.

Proof. The main ingredient (M. Barge and J. Martin, 1985): Since $f : [a, b] \to [a, b]$ is topologically transitive, then

(*i*) either f is mixing.

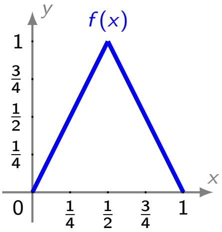
(*ii*) or $c \in (a, b)$ is the unique fixed point of f, $f([a, c]) = [c, b]$, $f([c, b]) = [a, c]$ and both maps $f^{\mathbf{2}}_{|[a,c]}$, $f^{\mathbf{2}}_{|[c,b]}$ are mixing.

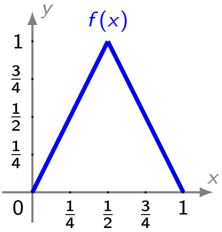
Combine this with [M. Murillo-Arcila and A. Peris, 2015] and the next lemma to prove that $\widehat{f^{\mathbf{2}}}$ is mixing.

Lemma

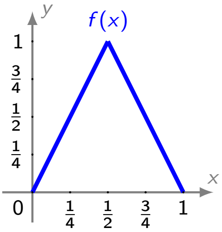
Assume that $T : X \to X$, $X = Y \oplus Z$ where Y and Z are invariant under T. If $T\restriction_Y$ and $T\restriction_Z$ are mixing, then so is T.

$\square$


y
$f(x)$
1
$\frac{3}{4}$
$\frac{1}{2}$
$\frac{1}{4}$
0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
x

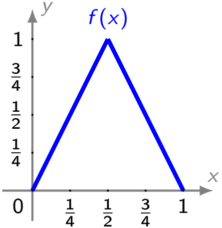


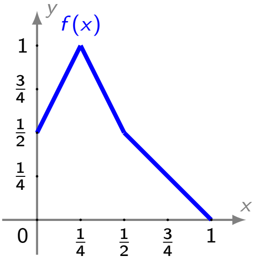
f is mixing and Devaney chaotic.



f is mixing and Devaney chaotic.

One can check that $\widehat{f}$ acts as a kind of backwards shift on the haar basis $(h_m)_m$ of $L^1([0,1])$.





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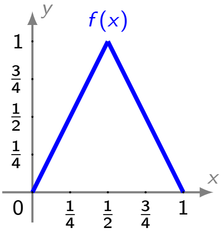

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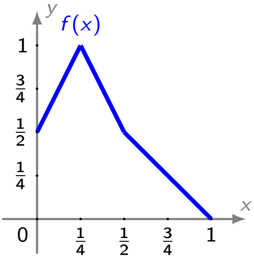
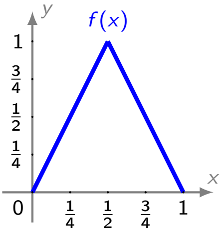


f is topologically transitive and Devaney chaotic, but not weakly mixing.



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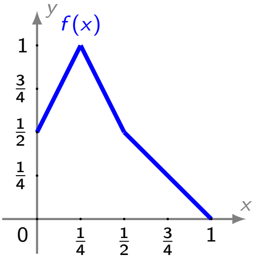
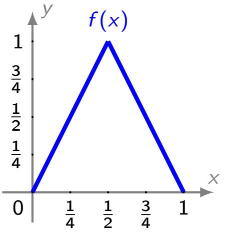


f is topologically transitive and Devaney chaotic, but not weakly mixing.

f^2 is not topologically transitive.



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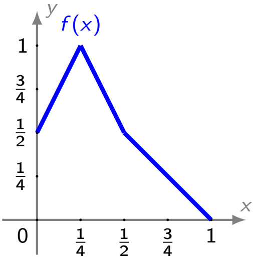

f is topologically transitive and Devaney chaotic, but not weakly mixing.

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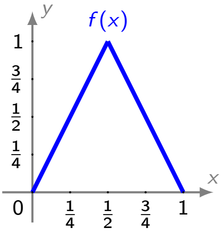
The operator $\widehat{f}$ is actually mixing.



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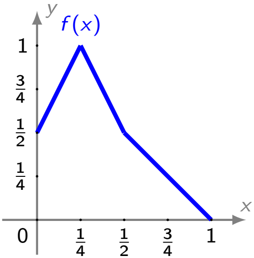

f is topologically transitive and Devaney chaotic, but not weakly mixing.

f^2 is not topologically transitive.

The operator $\widehat{f}$ is actually mixing.

$$\widehat{g} \text{ mixing } \not\Longrightarrow g \text{ transitive.}$$

Merci pour votre attention!

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In general:

$$
\begin{tikzcd}
\widehat{f}\ \text{mixing} \arrow[r, Rightarrow] & \widehat{f}\ \text{weakly mixing} \arrow[r, Leftrightarrow] & \widehat{f}\ \text{satisfies the HC} \arrow[d, Rightarrow, bend left, "\times"] \\
f\ \text{mixing} \arrow[r, Rightarrow] \arrow[u, Rightarrow, "{[MP]}"] & f\ \text{weakly mixing} \arrow[r, Rightarrow] \arrow[u, Rightarrow, "{[MP]}"] & \widehat{f}\ \text{satisfies the HCL} \arrow[u, Rightarrow, "\text{HCL}"]
\end{tikzcd}
$$

In general:

$$
\begin{tikzcd}
\widehat{f}\ \text{satisfies the HC} \arrow[r, Rightarrow] \arrow[rrd, Rightarrow, "\times"] & \widehat{f}\ \text{hypercyclic} \arrow[r, Rightarrow] \arrow[l, Rightarrow, bend right, "?"'] & \widehat{f}\ \text{supercyclic} \arrow[r, Rightarrow] \arrow[l, Rightarrow, bend right, "\times"'] & \widehat{f}\ \text{cyclic} \arrow[l, Rightarrow, bend right, "\times"'] \\
\widehat{f}\ \text{satisfies the HCL} \arrow[u, Rightarrow] \arrow[rr, Rightarrow, "\times"] & & f\ \text{has a dense orbit} \arrow[u, Rightarrow, "\times"] \arrow[ru, Rightarrow] &
\end{tikzcd}
$$

For $[a, b]$:

$$
\begin{tikzcd}
\widehat{f}\ \text{weakly mixing and chaotic} \arrow[rr, Leftrightarrow] & & \widehat{f}\ \text{weakly mixing} \\
f\ \text{weakly mixing} \arrow[r, Rightarrow] & f\ \text{Devaney chaotic} \arrow[r, Leftrightarrow, "{[VB]}"] \arrow[lu, Rightarrow] \arrow[l, Rightarrow, bend left, "\times"] & f\ \text{is transitive} \arrow[u, Rightarrow]
\end{tikzcd}
$$