Coarse embeddings of Kalton's interlaced graphs

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Joint work with G. Lancien and A. Procházka

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We equip $G_k(\mathbb{M})$ with the graph metric d satisfying $d(\overline{n}, \overline{m}) = 1$ whenever $\overline{n} \neq \overline{m}$ and

$$n_1 \leq m_1 \leq n_2 \ldots \leq n_k \leq m_k \text{ or } m_1 \leq n_1 \leq m_2 \ldots \leq m_k \leq n_k.$$

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For instance, in the following case :

$$\underbrace{n_1 < n_2 < \dots < n_{k_0}} < \underbrace{m_1 < m_2 < \dots < m_{k_0}} < n_{k_0+1} < m_{k_0+1} < \dots < n_k < m_k.$$

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4. The James space $\mathcal{J}$ and its dual $\mathcal{J}^*$ fail property $\mathcal{Q}$ ("not so easy", based on James sequences in non reflexive spaces).

We recall that :

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$\rightarrow$ This result isolate a coarse invariant which is close to but different from property $\mathcal{Q}$.

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$$\begin{aligned}
\overline{\rho}_X(t, x, Y) &= \sup_{y \in S_Y} \|x + ty\| - 1 \\
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Moreover, we will say that the modulus $\overline{\rho}_X$ is of power type $p \in (1,\infty)$ (in short p-(AUS)) if there is a constant $c > 0$ such that :

$$\forall t > 0,\ \overline{\rho}_X(t) \leq ct^p.$$

weak*-(AUC)

Similarly, there is in X^* a modulus of weak* asymptotic uniform convexity defined by

$$\overline{\delta}^*_X(t) = \inf_{x^* \in S_{X^*}} \sup_E \inf_{y^* \in S_E} \|x^* + ty^*\| - 1,$$

where E runs through all weak*-closed subspaces of X^* of finite codimension.

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Then, the norm $\|\cdot\|_{X^*}$ is said to be weak* *asymptotically uniformly convex* (in short weak* AUC) if

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Fact

The norm of X is p-(AUS) if and only if the norm of X^* is weak* q-(AUC) where q is the conjugate exponent of p.

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Then, $(G_k(\mathbb{N}))_{k \in \mathbb{N}}$ does not equi-coarsely embed into X^*.

The main result

We recall that a Banach space is said to be quasi-reflexive if $\dim(X^{**}/X) < \infty$.

Theorem (LPP, '18)

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The family $(G_k(\mathbb{N}))_{k\in\mathbb{N}}$ does not equi-coarsely embed into $\mathcal{J}$ and $\mathcal{J}^*$.

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The family $(G_k(\mathbb{N}))_{k\in\mathbb{N}}$ does not equi-coarsely embed into $\mathcal{J}$ and $\mathcal{J}^*$. The same results holds for $\mathcal{J}_p$ and $\mathcal{J}_p^*$.

Step 1 : weak*-null tree

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Lemma

For every Lipschitz map $f : G_k(\mathbb{N}) \to X^*$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and a weak*-null tree $(x^*(\overline{n}))_{\overline{n}\in[\mathbb{M}]^{\leq k}}$ in X^* such that

- $\|x^*(\overline{m})\|_{X^*} \leq \mathrm{Lip}(f)$, *for every* $\overline{m} \in [\mathbb{M}]^{\leq k} \setminus \{\emptyset\}$.
- $f(\overline{n}) = \sum_{i=0}^{n} x^*(n_1, \ldots, n_i)$, *for every* $\overline{n} \in G_k(\mathbb{M})$.

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Idea of the proof : Induction on $k \in \mathbb{N}$, using weak*-compactness and a diagonal argument.

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We decompose f as a weak*-null tree in X^{***} :

$$\forall \overline{n} \in G_k(\mathbb{N}) \ , \ f(\overline{n}) = \sum_{i=0}^{n} z(n_1, \dots, n_i) \in X^{***},$$
$$\text{with } \|z(n_1, \dots, n_i)\| \leq \omega(1) \text{ if } i \neq 0.$$

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We decompose $z(n_1, \dots, n_i)$ through the identification $X^{***} = X^* \oplus X^{\perp}$:

$$z(n_1, \dots, n_i) = \underbrace{x^*(n_1, \dots, n_i)}_{\in X^*} + \underbrace{t(n_1, \dots, n_i)}_{\in X^{\perp}},$$

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Remarks

1. Since $t(n_1, \ldots, n_i) \in X^\perp$, we get that $(x^*(\overline{n}))_{\overline{n} \in [\mathbb{N}]^{\leq k}}$ is a weak*-null tree in X^*.
2. Moreover, since f actually takes values in X^*, we have that :

$$\forall \overline{n} \in G_k(\mathbb{N}),\ f(\overline{n}) = \sum_{i=0}^{n} x^*(n_1, \ldots, n_i).$$

Step 2 : Using the weak* A. U. Convexity

Claim

Up to extract a full subgraph ("$G_k(\mathbb{M})$ instead of $\mathbb{G}_k(\mathbb{N})$"), we may assume the existence of $C_1 > 0$ such that the following holds :

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We write $E_r = \{\overline{n} = (n_1, \dots, n_j) \in [\mathbb{N}]^{\leq k} \ : \ n_j = r\}$.
Then, $\forall \overline{n}^1 \in E_{r_1}, \dots, \overline{n}^l \in E_{r_l}$ $(r_1 < \dots < r_l \in \mathbb{N})$, $\forall \varepsilon_1, \dots, \varepsilon_l \in \{-1, 1\}$:

$$\Big\| \sum_{i=1}^{l} \varepsilon_i x^*(\overline{n}^i) \Big\|_{X^*} \geq C_1 \Big(\sum_{i=1}^{l} \|x^*(\overline{n}^i)\|_{X^*}^2 \Big)^{1/2}.$$

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Idea of the proof : Let $|\cdot|$ be an equivalent weak* 2-(AUC) norm on X^* with modulus : $\overline{\delta}^*_{|\cdot|}(t) \geq 2c_1 t^2$. We let $\varphi_1(t) = c_1 t^2$

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$$\begin{aligned} \Big|\sum_{i=1}^{l} \varepsilon_i x^*(\overline{n}^i) + x^*(\overline{n}^{l+1})\Big| := |S + x^*| &\geq |S| + \frac{|S|}{2}\,\overline{\delta}^*_{|\cdot|}\Big(\frac{|x^*|}{|S|}\Big) \\ &\geq |S| + c_1|S|\frac{|x^*|^2}{|S|^2} := N_2^{\varphi_1}(|S|, |x^*|) \end{aligned}$$

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& \geq \quad N_2^{\varphi_1}\Big(N_l(|x^*(\overline{n}^1)|, \ldots, |x^*(\overline{n}^l)|), |x^*|\Big) \\
& = \quad N_{l+1}^{\varphi_1}(|x^*(\overline{n}^1)|, \ldots, |x^*(\overline{n}^{l+1})|)
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&\geq \quad C_1'\Big(\sum_{i=1}^{l+1} |x^*(\overline{n}^i)|^2_{X^*}\Big)^{1/2}.
\end{aligned}$$

Step 3 : "The interlaced argument"

Ramsey theorem : Again, up to extract a full subgraph, me may assume :

$$\forall j \leq k,\ \exists K_j,\ \forall n_1 < \ldots < n_j\ :\ \|x^*(n_1, \ldots, n_j)\| \simeq K_j.$$

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$$\begin{aligned}
\omega(1) \ge \|f(\overline{n}) - f(\overline{m})\|_{X^*} &= \Big\| \sum_{i=1}^{k} x^*(n_1, \ldots, n_i) - \sum_{i=1}^{k} x^*(m_1, \ldots, m_i) \Big\|_{X^*} \\
&\ge C_1 \Big(\sum_{i=1}^{k} \|x^*(n_1, \ldots, n_i)\|_{X^*}^2 + \|x^*(m_1, \ldots, m_i)\|_{X^*}^2 \Big)^{1/2} \\
&\ge C_1 \big(2 \sum_{i=1}^{k} K_j^2\big)^{1/2} \\
&\ge \sqrt{2} C_1 \Big(\sum_{i=1}^{k} K_j^2 \Big)^{1/2}.
\end{aligned}$$

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Since $\sum_{i=1}^{k} K_j^2 \leq \frac{\omega(1)^2}{2C_1^2}$, there exists $j \in \{0, N^2 - N\}$ such that :

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From now on, we start the construction of $\overline{n}, \overline{m}$ such that $d(\overline{n}, \overline{m}) = N$ but $\|f(\overline{n}) - f(\overline{m})\|_{X^*}$ is "too small" (in fact smaller than $\rho(N)$).

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- For $i \leq j$ let $n_i = m_i = i$
- For $j+1 \leq i \leq j+N$, let $n_i = i$ and $m_i = i + N$
- For $j+N+1 \leq i \leq k$, $n_i = m_i$ **"large enough", to be precised...**

$$\underbrace{n_1 = m_1 < \ldots < n_j = m_j}_{Equality} < \underbrace{n_{j+1} < \ldots < n_{j+N}}_{\text{Block of size N}} < \underbrace{m_{j+1} < \ldots < m_{j+N}}_{\text{Block of size N}}$$

$$< \underbrace{n_{j+N+1} = m_{j+N+1} < \ldots < n_k = m_k}_{Equality}.$$

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$$\begin{aligned} A &\leq 2\sum_{i=j+1}^{j+N} K_i \leq 2\sqrt{N}\Big(\sum_{i=j+1}^{j+N} K_i^2\Big)^{1/2} \\ &\leq \frac{2\sqrt{N}}{\sqrt{N}}\Big(\frac{\omega(1)^2}{2C_1^2}\Big)^{1/2} \\ &\leq \sqrt{2}\,\frac{\omega(1)}{C_1}. \end{aligned}$$

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In this final step, we finish the construction of $\overline{n}, \overline{m}$ in such a way that we manage to deal with B.

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We denote $|\cdot|$ a 2-(AUS) norm on X^*, with modulus

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Since $X^\perp$ is finite dimensional, we use Ramsey's theorem to stabilize the elements $t(n_1, \dots, n_i)$:

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Proposition (G. Lancien - M. Raja, 2017)

Let X be a Banach space. Then the bidual norm on X^{**} has the following property :

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$$\overline{\rho}_{|\cdot|}(t) \leq \frac{c_2}{2} t^2, \text{ for some } c_2 > 0. \text{ Let } \varphi_2(t) = c_2 t^2.$$

Since $X^\perp$ is finite dimensional, we use Ramsey's theorem to stabilize the elements $t(n_1, \ldots, n_i)$:

$$\forall i \leq k, \forall n_1 < \ldots < n_i \ : \ t(n_1, \ldots, n_i) \simeq t_i.$$

Proposition (G. Lancien - M. Raja, 2017)

Let X be a Banach space. Then the bidual norm on X^{**} has the following property :
For any $t \in (0,1)$, any weak*-null sequence $(x_n^{**})_{n=1}^{\infty}$ in $B_{X^{**}}$ and any $x \in S_X$ we have :

$$\limsup_{n \to \infty} \|x + t x_n^{**}\| \leq 1 + \overline{\rho}_X(t, x).$$

Step 4 : Using the A. U. Smoothness

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Using this last result, there exists $n_{j+N+2} = m_{j+N+2} > n_{j+N+1}$ such that the following holds :

$$\begin{aligned}
&|x^*(\dots, n_{j+N+1}) - x^*(\dots, m_{j+N+1}) + x^*(\dots, n_{j+N+2}) - x^*(\dots, m_{j+N+2})| \\
&\quad \simeq |x^*(\dots, n_{j+N+1}) - x^*(\dots, m_{j+N+1}) + z(\dots, n_{j+N+2}) - z(\dots, m_{j+N+2})| \\
&\quad := |x^* + z| \leq |x^*| + c_2|x^*| \frac{|z|^2}{|x^*|^2} = N_2^{\varphi_2}(|x^*|, |z|) \\
&\quad \simeq N_2^{\varphi_2}(|x^*(\dots, n_{j+N+1}) - x^*(\dots, m_{j+N+1})|, |x^*(\dots, n_{j+N+2}) - x^*(\dots, m_{j+N+2})|)
\end{aligned}$$

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$$
\begin{aligned}
&\Big| \sum_{i=j+1}^{j+N} x^*(n_1, \dots, n_i) - x^*(m_1, \dots, m_i) \Big| \\
&\qquad \leq N^{\varphi_2}_{K-j-N}\big(|x^*(\dots, n_{j+N+1} - x^*(\dots, m_{j+N+1})|, \dots, |x^*(\dots, n_k) - x^*(\dots, m_k)|\big) \\
&\qquad \leq N^{\varphi_2}_{k-j-N}(2K_{j+N+1}, \dots, 2K_k) \\
&\qquad \leq C_2' \Big(\sum_{i=j+N+1}^{k} 4\, K_i^2 \Big)^{1/2}.
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&\Big|\sum_{i=j+1}^{j+N} x^*(n_1,\dots,n_i) - x^*(m_1,\dots,m_i)\Big| \\
&\qquad \le N^{\varphi_2}_{K-j-N}\big(|x^*(\dots,n_{j+N+1} - x^*(\dots,m_{j+N+1})|,\dots,|x^*(\dots,n_k) - x^*(\dots,m_k)|\big) \\
&\qquad \le N^{\varphi_2}_{k-j-N}(2K_{j+N+1},\dots,2K_k) \\
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\end{aligned}$$

And so going back to the original norm :

$$B \le C_2\Big(\sum_{i=j+N+1}^{k} K_i^2\Big)^{1/2} \le \frac{C_2}{\sqrt{2}C_1}\omega(1).$$

Step 5 : Conclusion

Finally,

$$\|f(\overline{n}) - f(\overline{m})\|_{X^*} \leq A + B \leq \sqrt{2}\,\frac{\omega(1)}{C_1} + \frac{C_2}{\sqrt{2}C_1}\omega(1).$$

But $d(\overline{n}, \overline{m}) = N$, this contradicts the fact that

$$\rho(N) > \sqrt{2}\,\frac{\omega(1)}{C_1} + \frac{C_2}{\sqrt{2}C_1}\omega(1).$$

$\square$

Thank you very much !