Schur properties over some Lipschitz free-spaces

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Remark

$\delta_M : x \in M \mapsto \delta_M(x) \in \mathcal{F}(M)$ is a non linear isometry.

Proposition

The Lipschitz-free space $\mathcal{F}(M)$ has the following property :
$\forall X$ *Banach*, $\forall f: M \to X$ *Lipschitz*, $\exists! \overline{f}: \mathcal{F}(M) \to X$ *with* $\|\overline{f}\| = \|f\|_L$ *and such that the following diagram commutes*

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The map $f \in Lip_0(M, X) \mapsto \overline{f} \in \mathcal{L}(\mathcal{F}(M), X)$ *is an onto linear isometry.*

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$\mathcal{F}(M)^* = Lip_0(M)$. Uniqueness of the predual for : m. s. of finite diameter and complete and convex m. s. ($\Rightarrow$ Banach spaces) (Weaver, 2016)

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where δ_{M_i} is the isometry defined above :

$$\delta_{M_i} : \ x \in M_i \mapsto \delta_{M_i}(x) \in \mathcal{F}(M_i).$$

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Simple spaces ?Compact m. s., Proper m. s., Finite dimensional Banach spaces with any norm $\longrightarrow \ell_1$, c_0...

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- (Lancien - Pernecká) : For every $M \subseteq R^n$, the space $\mathcal{F}(M)$ has the bounded approximation property (BAP).

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ii) $lip_0(\mathbb{N}) = Lip_0(\mathbb{N})$, and also $lip_0(D) = Lip_0(D)$ for any uniformly discrete metric space D.

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We say that a subspace $S \subseteq Lip_0(M)$ separates points uniformly (S.P.U.) if there is a constant $C \geq 1$ such that $\forall x \neq y \in M$, $\forall \varepsilon > 0$, $\exists f \in S$ with $\|f\|_L \leq C + \varepsilon$ and $|f(x) - f(y)| = d(x, y)$.

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(Kalton) : $S \subseteq Lip_0(M)$ S.P.U. with constant C if and only if S is a C norming subspace of $Lip(M) = \mathcal{F}(M)^*$, that is :

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Proposition (Weaver/Dalet)

i) *Let (K, d) be a compact metric space then*

$$lip_0(K) \text{ S.P.U.} \Leftrightarrow lip_0(K)^* = \mathcal{F}(K).$$

ii) *Let (M, d) be a proper metric space then*

$$S_0(M) \text{ S.P.U.} \Leftrightarrow S_0(M)^* = \mathcal{F}(M).$$

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iv) (P.) : M some metric spaces originate from p-Banach spaces (to be specified in the last section).

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Let X be a Banach space. We say that X has the Schur property if : $\forall (x_n)_n \subset X$, $x_n \xrightarrow[n\to\infty]{\omega} 0 \Longrightarrow \|x_n\| \xrightarrow[n\to\infty]{} 0$. $(\omega = \sigma(X, X^*))$

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i) ℓ_1 has the Schur property. (gliding hump argument)
ii) Infinite dimensional reflexive spaces are not Schur spaces.

Proposition

If a Banach space X has the Schur property, then it contains ℓ_1 hereditary.

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i) (Kalton, 2004) : Consider $(M, \omega \circ d)$ where (M, d) is a pointed metric space and ω is a nontrivial gauge (typically $\omega(t) = t^p$ with $0 < p < 1$). Then $\mathcal{F}(M, \omega \circ d)$ has the Schur property.

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Proposition (P.)

Let (M, d) be a pointed metric space such that $lip_0(M)$ is 1-norming. Then $\mathcal{F}(M)$ has the Schur property.

Theorem (P.)

Let (M,d) be a proper pointed metric space. Assume that :

i) $S_0(M)$ *S.P.U.*

ii) $\mathcal{F}(M)$ *has* (MAP).

Then for every $\varepsilon > 0$, there exist $(E_n)_n$ a sequence of finite dimensional subspaces of $\mathcal{F}(M)$ such that $\mathcal{F}(M) \underset{1+\varepsilon}{\hookrightarrow} (\sum \oplus_n E_n)_{\ell_1}$.

Moreover the embedding is ω^* to ω^* continuous.

Lemma (Godefroy-Kalton-Li)

Let V be a subspace of c_0 with (MAP). Then for every $\varepsilon > 0$, there exist $(E_n)_n$ a sequence of finite dimensional subspaces of V^* and a ω^* to ω^* continuous linear map $T : V^* \to (\sum \oplus_n E_n)_{\ell_1}$ such that :

$$\forall x^* \in V^* : (1-\varepsilon)\|x^*\| \leq \|Tx^*\| \leq (1+\varepsilon)\|x^*\|.$$

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Theorem (Grothendieck)

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ii) Conversely, if X is a locally bounded topological vector space, we can define a quasi-norm on X : Pick B a bounded neighbourhood of 0 and consider the Minkowski functional of B : $\mu_B(x) = \inf\{\lambda \geq 0 : \lambda^{-1}x \in B\}$.

$\longrightarrow$ Quasi-Banach = Quasi-normed + complete.

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iv) (Aoki-Rolewicz) $\Longrightarrow$ Every quasi-normed space can be renormed to be a p-Banach.

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Proposition (P.)

Let X_p be a p-Banach space which admits a monotone FDD (ℓ_p for example). We denote $M_p = (X_p, d_p)$. Then $lip_0(M)$ is 1-norming and thus $\mathcal{F}(M_p)$ has the Schur property. Moreover $\mathcal{F}(M_p)$ has (MAP).

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