Norm attainment in spaces of vector valued Lipschitz functions

Colin PETITJEAN

Journées de l'école doctorale Carnot-Pasteur 2017 Dijon, May 19, 2017





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Definition

 $\begin{array}{l} \mathsf{Lipschitz-free space over } M:\\ \mathcal{F}(M):=\overline{\mathsf{span}\left\{\delta_M(x)\,;\,x\in M\right\}}^{\|\cdot\|}\subset \mathit{Lip}_0(M)^*. \end{array}$

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Lipschitz-free space over M: $\mathcal{F}(M) := \overline{\text{span} \{\delta_M(x); x \in M\}}^{\|\cdot\|} \subset Lip_0(M)^*.$

Remark

 $\delta_M : x \in M \mapsto \delta_M(x) \in \mathcal{F}(M)$ is a non linear isometry.

Proposition (Fundamental factorisation property)

The Lipschitz-free space $\mathcal{F}(M)$ has the following property : $\forall X \text{ Banach}, \forall f : M \to X \text{ Lipschitz}, \exists ! \overline{f} : \mathcal{F}(M) \to X \text{ with } \|\overline{f}\| = \|f\|_L$ and such that the following diagram commutes



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The map $f \in Lip_0(M, X) \mapsto \overline{f} \in \mathcal{L}(\mathcal{F}(M), X)$ is an onto linear isometry. We write $Lip_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X)$.

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Remark : For $X = \mathbb{R}$ we obtain : $Lip_0(M) \equiv \mathcal{F}(M)^*$.

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Explore the linear structure of $\mathcal{F}(M)$ for "simple spaces M". Simple spaces? Compact m. s., Proper m. s., Finite dimensional Banach spaces with any norm $\longrightarrow \ell_1, c_0...$ Motivation :

Nonlinear classification of Banach spaces

Little Lipschitz spaces and double duality result

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We say that $lip_0(M)$ separates points uniformly (SPU) if there is a constant $C \ge 1$ such that $\forall x \ne y \in M$, $\exists f \in lip_0(M)$ with $\|f\|_L \le C$ and |f(x) - f(y)| = d(x, y).

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Examples (Trivial)

i) $lip_0(\mathbb{R}) = \{0\}$, and also $lip_0(X) = \{0\}$ for any Banach space X.

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- ii) $lip_0(\mathbb{N}) = Lip_0(\mathbb{N})$, and also $lip_0(D) = Lip_0(D)$ for any uniformly discrete metric space D.

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- i) (Weaver) : M the middle-third Cantor set.
- ii) (Godefroy) : *M* "small" Cantor set.
- iii) (Dalet) : *M* countable compact.
- iv) "Many other families of examples..."

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$- \mathcal{NA}(X, Y) := \{ T \in \mathcal{L}(X, Y) : \exists x \in B_X, \|L(x)\|_Y = \|T\| \}.$

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Questions

Bearing in mind $Lip_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X)$, we wonder if the two previous notions of norm attainment are the same, and if there is an equivalent version of the Bishop-Phelps theorem in spaces of Lipschitz functions.

Scalar-valued case

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Theorem (García-Lirola, Rueda Zoca, P.)

Let *M* be a compact *m*. s. such that $lip_0(M)^* \equiv \mathcal{F}(M)$. Then $\mathcal{NA}(\mathcal{F}(M), \mathbb{R}) = Lip_{SNA}(M, \mathbb{R})$. Thus, according to Bishop-Phelps theorem, we have

$$\overline{Lip_{SNA}(M,\mathbb{R})}^{\|\cdot\|} = Lip_0(M).$$

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${\sf Ingredients} \ {\sf of} \ {\sf the} \ {\sf proof}:$

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Let *M* be a metric space and let $\gamma \in \text{ext}(B_{Lip_0(X)^*})$. Then, $\gamma \in \mathcal{F}(M)$ if and only if $\gamma = \frac{\delta_M(x) - \delta_M(y)}{d(x,y)}$ for some $x \neq y$.

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Lemma (Godefroy)

Let X be a Banach space which is an M-ideal in its bidual, that is $X^{***} = X^{\perp} \oplus_1 X^*$. If x^{**} attains its norm on B_{X^*} , then x^{**} attains its norm on some $x^* \in B_{X^*} \cap ext(B_{X^{***}})$.

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If M is compact, then for every $\varepsilon > 0$, there is $Z \subseteq c_0$ which is $(1 + \varepsilon)$ -isomorphic to $lip_0(M)$.

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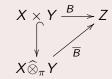
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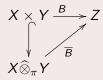
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Thus $\mathcal{B}(X \times Y, Z) \equiv \mathcal{L}(X \widehat{\otimes}_{\pi} Y, Z)$.

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Definition (Vector-valued Lipschitz-free space)

We may define the X-valued Lipschitz-free space over M to be : $\mathcal{F}(M)\widehat{\otimes}_{\pi}X$.

Vector valued case

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Theorem (García-Lirola, Rueda Zoca, P.)

Let M be a proper m. s. such that $lip_0(M)^* \equiv \mathcal{F}(M)$. Assume that Y^* has (RNP), and $\mathcal{F}(M)$ or Y^* has (AP), then : $\overline{\mathcal{NA}(\mathcal{F}(M), Y^{**})}^{\|\cdot\|} = \mathcal{L}(\mathcal{F}(M), Y^{**})$ and $\overline{Lip_{SNA}(M, Y)}^{\|\cdot\|} = Lip_0(M, Y^{**}).$

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Caution ! No Bishop-Phelps theorem in the vector valued case.

Theorem (García-Lirola, Rueda Zoca, P.)

Let M be a proper m. s. such that $lip_0(M)^* \equiv \mathcal{F}(M)$. Assume that Y^* has (RNP), and $\mathcal{F}(M)$ or Y^* has (AP), then : $\overline{\mathcal{NA}(\mathcal{F}(M), Y^{**})}^{\|\cdot\|} = \mathcal{L}(\mathcal{F}(M), Y^{**})$ and $\overline{Lip_{SNA}(M, Y)}^{\|\cdot\|} = Lip_0(M, Y^{**}).$

Main ideas : $\mathcal{L}(\mathcal{F}(M), Y^{**}) = (\mathcal{F}(M)\widehat{\otimes}_{\pi}Y^{*})^{*}$ and $(RNP) \Longrightarrow$ Krein Milman property \Longrightarrow Lot of extreme points.

Thank you very much!

Thank you very much ! Good luck to the other phd students !

- C. Petitjean, Lipschitz-free spaces and Schur properties, J. of Math. Anal. Appl. Available at : https://arxiv.org/abs/1603.01391
- L. García-Lirola, C. Petitjean and A. Rueda Zoca, *On the structure of spaces of vector-valued Lipschitz functions*, to appear in Studia Math. Available at : https://arxiv.org/pdf/1606.05999.pdf.