Norm attainment in spaces of vector valued Lipschitz functions

Colin PETITJEAN

Journées de l'école doctorale Carnot-Pasteur 2017
Dijon, May 19, 2017

ECOLE DOCTORALE
CARNOT - PASTEUR

UBFC

(M, d) and (N, d) pointed metric space with origin 0.

(M, d) and (N, d) pointed metric space with origin 0.
X, Y, Z real Banach spaces.

(M, d) and (N, d) pointed metric space with origin 0.
X, Y, Z real Banach spaces.
$Lip_0(M) = \{f : M \to \mathbb{R} \text{ Lipschitz } : f(0) = 0\}$

(M, d) and (N, d) pointed metric space with origin 0.
X, Y, Z real Banach spaces.
$Lip_0(M) = \{f : M \to \mathbb{R} \text{ Lipschitz } : f(0) = 0\}$

$$\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)}$$ *(Best Lipschitz constant of f)*

(M, d) and (N, d) pointed metric space with origin 0.
X, Y, Z real Banach spaces.
$Lip_0(M) = \{f : M \to \mathbb{R} \text{ Lipschitz } : f(0) = 0\}$

$$\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)}$$ *(Best Lipschitz constant of f)*

$(Lip_0(M), \|\cdot\|_L)$ Banach space.

(M, d) and (N, d) pointed metric space with origin 0.
X, Y, Z real Banach spaces.
$Lip_0(M) = \{f : M \to \mathbb{R} \text{ Lipschitz } : f(0) = 0\}$

$$\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x,y)} \quad \textit{(Best Lipschitz constant of } f\textit{)}$$

$(Lip_0(M), \|\cdot\|_L)$ Banach space.

For $x \in M$, define $\delta_M(x) \in Lip_0(M)^*$ by $\langle \delta_M(x), f \rangle = f(x)$.

(M, d) and (N, d) pointed metric space with origin 0.
X, Y, Z real Banach spaces.
$Lip_0(M) = \{f : M \to \mathbb{R} \text{ Lipschitz } : f(0) = 0\}$

$$\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)} \quad \textit{(Best Lipschitz constant of } f\textit{)}$$

$(Lip_0(M), \|\cdot\|_L)$ Banach space.

For $x \in M$, define $\delta_M(x) \in Lip_0(M)^*$ by $\langle \delta_M(x), f \rangle = f(x)$.

Definition

Lipschitz-free space over M :
$\mathcal{F}(M) := \overline{\text{span}\,\{\delta_M(x)\,;\, x \in M\}}^{\|\cdot\|} \subset Lip_0(M)^*$.

(M, d) and (N, d) pointed metric space with origin 0.
X, Y, Z real Banach spaces.
$Lip_0(M) = \{f : M \to \mathbb{R} \text{ Lipschitz } : f(0) = 0\}$

$$\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x,y)} \quad \textit{(Best Lipschitz constant of } f\textit{)}$$

$(Lip_0(M), \|\cdot\|_L)$ Banach space.

For $x \in M$, define $\delta_M(x) \in Lip_0(M)^*$ by $\langle \delta_M(x), f \rangle = f(x)$.

Definition

Lipschitz-free space over M :
$\mathcal{F}(M) := \overline{\text{span}\{\delta_M(x)\,;\, x \in M\}}^{\|\cdot\|} \subset Lip_0(M)^*$.

Remark

$\delta_M : x \in M \mapsto \delta_M(x) \in \mathcal{F}(M)$ is a non linear isometry.

Proposition (Fundamental factorisation property)

The Lipschitz-free space $\mathcal{F}(M)$ has the following property :
$\forall X$ *Banach*, $\forall f : M \to X$ *Lipschitz*, $\exists! \overline{f} : \mathcal{F}(M) \to X$ *with* $\|\overline{f}\| = \|f\|_L$ *and such that the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "\delta_M"'] & X \\ \mathcal{F}(M) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Proposition (Fundamental factorisation property)

The Lipschitz-free space $\mathcal{F}(M)$ has the following property :
$\forall X$ *Banach*, $\forall f : M \to X$ *Lipschitz*, $\exists! \overline{f} : \mathcal{F}(M) \to X$ *with* $\|\overline{f}\| = \|f\|_L$ *and such that the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "\delta_M"'] & X \\ \mathcal{F}(M) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

The map $f \in Lip_0(M, X) \mapsto \overline{f} \in \mathcal{L}(\mathcal{F}(M), X)$ *is an onto linear isometry. We write* $Lip_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X)$.

Proposition (Fundamental factorisation property)

The Lipschitz-free space $\mathcal{F}(M)$ has the following property :
$\forall X$ *Banach*, $\forall f : M \to X$ *Lipschitz*, $\exists! \overline{f} : \mathcal{F}(M) \to X$ *with* $\|\overline{f}\| = \|f\|_L$ *and such that the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "\delta_M"'] & X \\ \mathcal{F}(M) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

The map $f \in Lip_0(M, X) \mapsto \overline{f} \in \mathcal{L}(\mathcal{F}(M), X)$ *is an onto linear isometry. We write* $Lip_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X)$.

Remark : For $X = \mathbb{R}$ we obtain : $Lip_0(M) \equiv \mathcal{F}(M)^*$.

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

ii) $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

ii) $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

iii) $\mathcal{F}(\mathbb{R}^2)$? $\mathcal{F}(\mathbb{R}^3)$?

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

ii) $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

iii) $\mathcal{F}(\mathbb{R}^2)$? $\mathcal{F}(\mathbb{R}^3)$?

Godefroy - Kalton program :

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

ii) $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

iii) $\mathcal{F}(\mathbb{R}^2)$? $\mathcal{F}(\mathbb{R}^3)$?

Godefroy - Kalton program :

Explore the linear structure of $\mathcal{F}(M)$ for "simple spaces M".

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

ii) $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

iii) $\mathcal{F}(\mathbb{R}^2)$? $\mathcal{F}(\mathbb{R}^3)$?

Godefroy - Kalton program :

Explore the linear structure of $\mathcal{F}(M)$ for "simple spaces M".

Simple spaces ?

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

ii) $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

iii) $\mathcal{F}(\mathbb{R}^2)$? $\mathcal{F}(\mathbb{R}^3)$?

Godefroy - Kalton program :

Explore the linear structure of $\mathcal{F}(M)$ for "simple spaces M".
Simple spaces ? Compact m. s., Proper m. s., Finite dimensional Banach spaces with any norm $\longrightarrow \ell_1$, c_0...

Examples

i) $\mathcal{F}(\mathbb{N}) \equiv \ell_1(\mathbb{N})$

ii) $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

iii) $\mathcal{F}(\mathbb{R}^2)$? $\mathcal{F}(\mathbb{R}^3)$?

Godefroy - Kalton program :

Explore the linear structure of $\mathcal{F}(M)$ for "simple spaces M".
Simple spaces ? Compact m. s., Proper m. s., Finite dimensional Banach spaces with any norm $\longrightarrow \ell_1$, c_0...

Motivation :

Nonlinear classification of Banach spaces

Definition

Let (M, d) be a metric space. We define the following closed subspace of $Lip_0(M)$:

Definition

Let (M,d) be a metric space. We define the following closed subspace of $Lip_0(M)$:

$$lip_0(M) := \left\{ f \in Lip_0(M) : \lim_{\varepsilon \to 0} \sup_{0<d(x,y)<\varepsilon} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \right\}.$$

Definition

Let (M,d) be a metric space. We define the following closed subspace of $Lip_0(M)$:

$$lip_0(M) := \left\{ f \in Lip_0(M) : \lim_{\varepsilon \to 0} \sup_{0<d(x,y)<\varepsilon} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \right\}.$$

We say that $lip_0(M)$ separates points uniformly (SPU) if there is a constant $C \geq 1$ such that $\forall x \neq y \in M$, $\exists f \in lip_0(M)$ with $\|f\|_L \leq C$ and $|f(x)-f(y)| = d(x,y)$.

Definition

Let (M, d) be a metric space. We define the following closed subspace of $Lip_0(M)$:

$$lip_0(M) := \left\{ f \in Lip_0(M) : \lim_{\varepsilon \to 0} \sup_{0<d(x,y)<\varepsilon} \frac{|f(x) - f(y)|}{d(x,y)} = 0 \right\}.$$

We say that $lip_0(M)$ separates points uniformly (SPU) if there is a constant $C \geq 1$ such that $\forall x \neq y \in M$, $\exists f \in lip_0(M)$ with $\|f\|_L \leq C$ and $|f(x) - f(y)| = d(x, y)$.

Examples (Trivial)

i) $lip_0(\mathbb{R}) = \{0\}$, and also $lip_0(X) = \{0\}$ for any Banach space X.

Definition

Let (M, d) be a metric space. We define the following closed subspace of $Lip_0(M)$:

$$lip_0(M) := \left\{ f \in Lip_0(M) : \lim_{\varepsilon \to 0} \sup_{0 < d(x,y) < \varepsilon} \frac{|f(x) - f(y)|}{d(x,y)} = 0 \right\}.$$

We say that $lip_0(M)$ separates points uniformly (SPU) if there is a constant $C \geq 1$ such that $\forall x \neq y \in M$, $\exists f \in lip_0(M)$ with $\|f\|_L \leq C$ and $|f(x) - f(y)| = d(x, y)$.

Examples (Trivial)

i) $lip_0(\mathbb{R}) = \{0\}$, and also $lip_0(X) = \{0\}$ for any Banach space X.

ii) $lip_0(\mathbb{N}) = Lip_0(\mathbb{N})$, and also $lip_0(D) = Lip_0(D)$ for any uniformly discrete metric space D.

Proposition (Weaver)

Let (M, d) be a compact metric space then

$$lip_0(M) \ S.P.U. \Leftrightarrow lip_0(M)^* = \mathcal{F}(M).$$

Proposition (Weaver)

Let (M, d) be a compact metric space then

$$lip_0(M) \ S.P.U. \Leftrightarrow lip_0(M)^* = \mathcal{F}(M).$$

(P. $\Longrightarrow$ In this case, $\mathcal{F}(M)$ has the Schur property and sometimes even better)

Proposition (Weaver)

Let (M,d) be a compact metric space then

$$lip_0(M)\ S.P.U. \Leftrightarrow lip_0(M)^* = \mathcal{F}(M).$$

(P. $\Longrightarrow$ In this case, $\mathcal{F}(M)$ has the Schur property and sometimes even better)

Examples (Nontrivial)

For M as follows, $lip_0(M)$ SPU :

Proposition (Weaver)

Let (M, d) be a compact metric space then

$$lip_0(M)\ S.P.U. \Leftrightarrow lip_0(M)^* = \mathcal{F}(M).$$

(P. $\Longrightarrow$ In this case, $\mathcal{F}(M)$ has the Schur property and sometimes even better)

Examples (Nontrivial)

For M as follows, $lip_0(M)$ SPU :

i) (Weaver) : M the middle-third Cantor set.

Proposition (Weaver)

Let (M, d) be a compact metric space then

$$lip_0(M)\ S.P.U. \Leftrightarrow lip_0(M)^* = \mathcal{F}(M).$$

(P. $\Longrightarrow$ In this case, $\mathcal{F}(M)$ has the Schur property and sometimes even better)

Examples (Nontrivial)

For M as follows, $lip_0(M)$ SPU :

i) (Weaver) : M the middle-third Cantor set.

ii) (Godefroy) : M "small" Cantor set.

Proposition (Weaver)

Let (M, d) be a compact metric space then

$$lip_0(M) \text{ S.P.U.} \Leftrightarrow lip_0(M)^* = \mathcal{F}(M).$$

(P. $\Longrightarrow$ In this case, $\mathcal{F}(M)$ has the Schur property and sometimes even better)

Examples (Nontrivial)

For M as follows, $lip_0(M)$ SPU :

i) (Weaver) : M the middle-third Cantor set.

ii) (Godefroy) : M "small" Cantor set.

iii) (Dalet) : M countable compact.

Proposition (Weaver)

Let (M, d) be a compact metric space then

$$lip_0(M) \ S.P.U. \Leftrightarrow lip_0(M)^* = \mathcal{F}(M).$$

(P. $\Longrightarrow$ In this case, $\mathcal{F}(M)$ has the Schur property and sometimes even better)

Examples (Nontrivial)

For M as follows, $lip_0(M)$ SPU :

i) (Weaver) : M the middle-third Cantor set.

ii) (Godefroy) : M "small" Cantor set.

iii) (Dalet) : M countable compact.

iv) “Many other families of examples..."

Definition

- $\mathcal{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \,:\, \exists x \in B_X,\ \|L(x)\|_Y = \|T\|\}.$

Definition

- $\mathcal{N}\mathcal{A}(X,Y) := \{T \in \mathcal{L}(X,Y)\ :\ \exists x \in B_X,\ \|L(x)\|_Y = \|T\|\}.$
- $Lip_{SNA}(M,Y) := \{f \in Lip_0(M,Y)\ :\ \exists x \neq y \in M,$
$$f(x) - f(y) = \|f\|_Y d(x,y)\}.$$

Definition

- $\mathcal{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \,:\, \exists x \in B_X,\ \|L(x)\|_Y = \|T\|\}$.
- $Lip_{SNA}(M,Y) := \{f \in Lip_0(M,Y) \,:\, \exists x \neq y \in M,$
$f(x) - f(y) = \|f\|_Y d(x,y)\}$.

Theorem (Bishop - Phelps)

$$\overline{\mathcal{NA}(X,\mathbb{R})}^{\|\cdot\|} = \mathcal{L}(X,\mathbb{R}) = X^*$$

Definition

- $\mathcal{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \,:\, \exists x \in B_X,\ \|L(x)\|_Y = \|T\|\}$.
- $Lip_{SNA}(M,Y) := \{f \in Lip_0(M,Y) \,:\, \exists x \neq y \in M,$
  $f(x) - f(y) = \|f\|_Y d(x,y)\}$.

Theorem (Bishop - Phelps)

$$\overline{\mathcal{NA}(X,\mathbb{R})}^{\|\cdot\|} = \mathcal{L}(X,\mathbb{R}) = X^*$$

Questions

Bearing in mind $Lip_0(M,X) \equiv \mathcal{L}(\mathcal{F}(M),X)$, we wonder if the two previous notions of norm attainment are the same, and if there is an equivalent version of the Bishop-Phelps theorem in spaces of Lipschitz functions.

Theorem (García-Lirola, Rueda Zoca, P.)

Let M be a compact m. s. such that $lip_0(M)^* \equiv \mathcal{F}(M)$. Then $\mathcal{N}\mathcal{A}(\mathcal{F}(M),\mathbb{R}) = Lip_{SNA}(M,\mathbb{R})$. Thus, according to Bishop-Phelps theorem, we have

$$\overline{Lip_{SNA}(M,\mathbb{R})}^{\|\cdot\|} = Lip_0(M).$$

Ingredients of the proof :

Ingredients of the proof :

Proposition (Weaver)

Let M be a metric space and let $\gamma \in ext(B_{Lip_0(X)^*})$. Then, $\gamma \in \mathcal{F}(M)$ if and only if $\gamma = \frac{\delta_M(x)-\delta_M(y)}{d(x,y)}$ for some $x \neq y$.

Ingredients of the proof :

Proposition (Weaver)

Let M be a metric space and let $\gamma \in ext(B_{Lip_0(X)^*})$. Then, $\gamma \in \mathcal{F}(M)$ if and only if $\gamma = \frac{\delta_M(x)-\delta_M(y)}{d(x,y)}$ for some $x \neq y$.

Lemma (Godefroy)

Let X be a Banach space which is an M-ideal in its bidual, that is $X^{***} = X^\perp \oplus_1 X^*$. If x^{**} attains its norm on B_{X^*}, then x^{**} attains its norm on some $x^* \in B_{X^*} \cap ext(B_{X^{***}})$.

Ingredients of the proof :

Proposition (Weaver)

Let M be a metric space and let $\gamma \in ext(B_{Lip_0(X)^*})$. Then, $\gamma \in \mathcal{F}(M)$ if and only if $\gamma = \frac{\delta_M(x)-\delta_M(y)}{d(x,y)}$ for some $x \neq y$.

Lemma (Godefroy)

Let X be a Banach space which is an M-ideal in its bidual, that is $X^{***} = X^{\perp} \oplus_1 X^*$. If x^{**} attains its norm on B_{X^*}, then x^{**} attains its norm on some $x^* \in B_{X^*} \cap ext(B_{X^{***}})$.

Lemma (Kalton)

If M is compact, then for every $\varepsilon > 0$, there is $Z \subseteq c_0$ which is $(1+\varepsilon)$-isomorphic to $lip_0(M)$.

Ingredients of the proof :

Proposition (Weaver)

Let M be a metric space and let $\gamma \in ext(B_{Lip_0(X)^*})$. Then, $\gamma \in \mathcal{F}(M)$ if and only if $\gamma = \frac{\delta_M(x)-\delta_M(y)}{d(x,y)}$ for some $x \neq y$.

Lemma (Godefroy)

Let X be a Banach space which is an M-ideal in its bidual, that is $X^{***} = X^{\perp} \oplus_1 X^*$. If x^{**} attains its norm on B_{X^*}, then x^{**} attains its norm on some $x^* \in B_{X^*} \cap ext(B_{X^{***}})$.

Lemma (Kalton)

If M is compact, then for every $\varepsilon > 0$, there is $Z \subseteq c_0$ which is $(1+\varepsilon)$-isomorphic to $lip_0(M)$.

For $x \in X$ and $y \in Y$, define $x \otimes y \in \mathcal{B}(X \times Y, Z)^*$ by :
$\langle x \otimes y, B \rangle = B(x, y)$.

For $x \in X$ and $y \in Y$, define $x \otimes y \in \mathcal{B}(X \times Y, Z)^*$ by :
$\langle x \otimes y, B \rangle = B(x, y)$.
Now let :
$X \widehat{\otimes}_\pi Y = \overline{\text{span}}^{\|\cdot\|}\{x \otimes y \,:\, x \in X,\, y \in Y\} \subseteq \mathcal{B}(X \times Y, Z)^*$.

For $x \in X$ and $y \in Y$, define $x \otimes y \in \mathcal{B}(X \times Y, Z)^*$ by :
$\langle x \otimes y, B \rangle = B(x, y)$.
Now let :
$X \widehat{\otimes}_\pi Y = \overline{\mathrm{span}}^{\|\cdot\|}\{x \otimes y \, : \, x \in X, \, y \in Y\} \subseteq \mathcal{B}(X \times Y, Z)^*$.

Proposition (Fundamental linearisation property)

Consider B in $\mathcal{B}(X \times Y, Z)$. Then there exists a unique continuous linear operator $\bar{B} : X \widehat{\otimes}_\pi Y \to Z$ such that $\|\bar{B}\| = \|B\|$ and such that the following diagram commutes

For $x \in X$ and $y \in Y$, define $x \otimes y \in \mathcal{B}(X \times Y, Z)^*$ by :
$\langle x \otimes y, B \rangle = B(x, y)$.
Now let :
$X \widehat{\otimes}_\pi Y = \overline{\mathrm{span}}^{\|\cdot\|}\{x \otimes y \,:\, x \in X,\, y \in Y\} \subseteq \mathcal{B}(X \times Y, Z)^*$.

Proposition (Fundamental linearisation property)

Consider B in $\mathcal{B}(X \times Y, Z)$. Then there exists a unique continuous linear operator $\overline{B} : X \widehat{\otimes}_\pi Y \to Z$ such that $\|\overline{B}\| = \|B\|$ and such that the following diagram commutes

$$\begin{tikzcd} X \times Y \arrow[r, "B"] \arrow[d, hook] & Z \\ X \widehat{\otimes}_\pi Y \arrow[ur, "\overline{B}"'] & \end{tikzcd}$$

For $x \in X$ and $y \in Y$, define $x \otimes y \in \mathcal{B}(X \times Y, Z)^*$ by :
$\langle x \otimes y, B \rangle = B(x, y)$.
Now let :
$X \widehat{\otimes}_\pi Y = \overline{\text{span}}^{\|\cdot\|}\{x \otimes y \,:\, x \in X,\, y \in Y\} \subseteq \mathcal{B}(X \times Y, Z)^*$.

Proposition (Fundamental linearisation property)

Consider B in $\mathcal{B}(X \times Y, Z)$. Then there exists a unique continuous linear operator $\bar{B} : X \widehat{\otimes}_\pi Y \to Z$ such that $\|\bar{B}\| = \|B\|$ and such that the following diagram commutes

$$\begin{tikzcd}
X \times Y \arrow[r, "B"] \arrow[d, hook] & Z \\
X \widehat{\otimes}_\pi Y \arrow[ur, "\overline{B}"'] &
\end{tikzcd}$$

Thus $\mathcal{B}(X \times Y, Z) \equiv \mathcal{L}(X \widehat{\otimes}_\pi Y, Z)$.

Remarks

i) $Z = \mathbb{R} : (X \widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

Remarks

i) $Z = \mathbb{R} : (X \widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

ii) It is easy to see that : $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$.

Remarks

i) $Z = \mathbb{R} : (X \widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

ii) It is easy to see that : $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$.

Thus we have $\mathcal{L}(X, Y^*) \equiv (X \widehat{\otimes}_\pi Y)^*$.

Remarks

i) $Z = \mathbb{R} : (X\widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

ii) It is easy to see that : $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$.

Thus we have $\mathcal{L}(X, Y^*) \equiv (X\widehat{\otimes}_\pi Y)^*$.
Recall that $Lip_0(M, X^*) \equiv \mathcal{L}(\mathcal{F}(M), X^*)$.

Remarks

i) $Z = \mathbb{R} : (X\widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

ii) It is easy to see that : $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$.

Thus we have $\mathcal{L}(X, Y^*) \equiv (X\widehat{\otimes}_\pi Y)^*$.
Recall that $Lip_0(M, X^*) \equiv \mathcal{L}(\mathcal{F}(M), X^*)$.
Finally we obtain $Lip_0(M, X^*) \equiv (\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$.

Remarks

i) $Z = \mathbb{R} : (X \widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

ii) It is easy to see that : $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$.

Thus we have $\mathcal{L}(X, Y^*) \equiv (X \widehat{\otimes}_\pi Y)^*$.
Recall that $Lip_0(M, X^*) \equiv \mathcal{L}(\mathcal{F}(M), X^*)$.
Finally we obtain $Lip_0(M, X^*) \equiv (\mathcal{F}(M) \widehat{\otimes}_\pi X)^*$.
This leads us to the following definition.

Remarks

i) $Z = \mathbb{R}$: $(X\widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

ii) It is easy to see that : $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$.

Thus we have $\mathcal{L}(X, Y^*) \equiv (X\widehat{\otimes}_\pi Y)^*$.
Recall that $Lip_0(M, X^*) \equiv \mathcal{L}(\mathcal{F}(M), X^*)$.
Finally we obtain $Lip_0(M, X^*) \equiv (\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$.
This leads us to the following definition.

Definition (Vector-valued Lipschitz-free space)

We may define the X-valued Lipschitz-free space over M to be : $\mathcal{F}(M)\widehat{\otimes}_\pi X$.

Proposition

Let M be a proper m. s. such that $S_0(M)^* \equiv \mathcal{F}(M)$. Then $\mathcal{NA}(\mathcal{F}(M), Y) = Lip_{SNA}(M, Y)$.

Proposition

Let M be a proper m. s. such that $S_0(M)^* \equiv \mathcal{F}(M)$. Then $\mathcal{NA}(\mathcal{F}(M), Y) = Lip_{SNA}(M, Y)$.

Caution ! No Bishop-Phelps theorem in the vector valued case.

Proposition

Let M be a proper m. s. such that $S_0(M)^* \equiv \mathcal{F}(M)$. Then $\mathcal{NA}(\mathcal{F}(M), Y) = Lip_{SNA}(M, Y)$.

Caution ! No Bishop-Phelps theorem in the vector valued case.

Theorem (García-Lirola, Rueda Zoca, P.)

Let M be a proper m. s. such that $lip_0(M)^* \equiv \mathcal{F}(M)$. Assume that Y^* has (RNP), and $\mathcal{F}(M)$ or Y^* has (AP), then :
$\overline{\mathcal{NA}(\mathcal{F}(M), Y^{**})}^{\|\cdot\|} = \mathcal{L}(\mathcal{F}(M), Y^{**})$ *and*
$\overline{Lip_{SNA}(M, Y)}^{\|\cdot\|} = Lip_0(M, Y^{**})$.

Proposition

Let M be a proper m. s. such that $S_0(M)^* \equiv \mathcal{F}(M)$. Then $\mathcal{NA}(\mathcal{F}(M), Y) = Lip_{SNA}(M, Y)$.

Caution ! No Bishop-Phelps theorem in the vector valued case.

Theorem (García-Lirola, Rueda Zoca, P.)

Let M be a proper m. s. such that $lip_0(M)^* \equiv \mathcal{F}(M)$. Assume that Y^* has (RNP), and $\mathcal{F}(M)$ or Y^* has (AP), then :
$\overline{\mathcal{NA}(\mathcal{F}(M), Y^{**})}^{\|\cdot\|} = \mathcal{L}(\mathcal{F}(M), Y^{**})$ *and*
$\overline{Lip_{SNA}(M, Y)}^{\|\cdot\|} = Lip_0(M, Y^{**})$.

Main ideas : $\mathcal{L}(\mathcal{F}(M), Y^{**}) = (\mathcal{F}(M)\widehat{\otimes}_\pi Y^*)^*$ and
$(RNP) \implies$ Krein Milman property $\implies$ Lot of extreme points.

Thank you very much !

Thank you very much !

Good luck to the other phd students !

C. Petitjean, *Lipschitz-free spaces and Schur properties*, J. of Math. Anal. Appl. Available at : https://arxiv.org/abs/1603.01391

L. García-Lirola, C. Petitjean and A. Rueda Zoca, *On the structure of spaces of vector-valued Lipschitz functions*, to appear in Studia Math. Available at : https://arxiv.org/pdf/1606.05999.pdf.