Some aspects of the structure of Lipschitz-free spaces and vector-valued Lipschitz functions

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Seminario Optimization y Equilibrio, CMM
Santiago de Chile, May 3, 2017

1. Notation
2. Lipschitz-free spaces
   - Definition and first properties
   - Little Lipschitz spaces and double duality results
   - Around some ℓ_1 properties
3. Vector valued case
   - Projective tensor product
   - Injective tensor product and bi-duality results
   - Natural questions
   - Norm attainment

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$X \hookrightarrow Y$: $\exists$ $Z \subseteq Y$ such that $X \simeq Z$.

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Remark

$\delta_M : x \in M \mapsto \delta_M(x) \in \mathcal{F}(M)$ is a non linear isometry.

Proposition (Fundamental factorisation property)

The Lipschitz-free space $\mathcal{F}(M)$ has the following property :
$\forall\, X$ *Banach*, $\forall\, f : M \to X$ *Lipschitz*, $\exists!\, \overline{f} : \mathcal{F}(M) \to X$ *with* $\|\overline{f}\| = \|f\|_L$ *and such that the following diagram commutes*

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The map $f \in Lip_0(M,X) \mapsto \overline{f} \in \mathcal{L}(\mathcal{F}(M), X)$ *is an onto linear isometry.*

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iii) ω^* *topo on bounded sets of* $Lip_0(M)$ = *topo of pointwise convergence.*

iv) *If* $\mu = \sum_{i=1}^n a_i\delta_M(x_i)$, $a_i \geq 0$ *with* $\sum_i a_i = 1$ *and* $\nu = \sum_{j=1}^m b_j\delta_M(y_j)$, $b_j \geq 0$ *with* $\sum_j b_j = 1$, *then*

$$\begin{aligned} \|\mu - \nu\| &= d_{OT}(\mu,\nu) \\ &= \inf\{\sum_{i,j} a_{ij} d(x_i,y_j) : \sum_j a_{ij} = a_i \sum_i a_{ij} = b_j\} \end{aligned}$$

($\to$ *Wasserstein distance, Kantorovich-Rubinstein theorem.*)

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Simple spaces ? Compact m. s., Proper m. s., Finite dimensional Banach spaces with any norm $\longrightarrow \ell_1$, c_0...

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ii) $lip_0(\mathbb{N}) = Lip_0(\mathbb{N})$, and also $lip_0(D) = Lip_0(D)$ for any uniformly discrete metric space D.

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We say that a subspace $S \subseteq Lip_0(M)$ separates points uniformly (S.P.U.) if there is a constant $C \geq 1$ such that $\forall x \neq y \in M$, $\forall \varepsilon > 0$, $\exists f \in S$ with $\|f\|_L \leq C + \varepsilon$ and $|f(x) - f(y)| = d(x, y)$.

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Proposition (Weaver/Dalet)

i) *Let (K,d) be a compact metric space then*

$$lip_0(K) \text{ S.P.U.} \Leftrightarrow lip_0(K)^* = \mathcal{F}(K).$$

ii) *Let (M,d) be a proper metric space then*

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v) (P.) : $M = (X, \|\cdot\|_p^p)$ metric space originating from a p-Banach spaces which admits a monotone FDD. ($0 < p < 1$, for instance $X = \ell_p$).

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Let X be a Banach space. We say that X has the Schur property if : $\forall (x_n)_n \subset X$, $x_n \xrightarrow[n\to\infty]{\omega} 0 \Longrightarrow \|x_n\| \xrightarrow[n\to\infty]{} 0$. $(\omega = \sigma(X, X^*))$

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ii) *$S_0(M)$ is 1-norming + M proper $\Longrightarrow \mathcal{F}(M)$ has the 1-strong Schur property.*

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Remark : There exist a compact countable metric space K such that $F(K)$ does not embed into ℓ_1.

Lemma (Godefroy-Kalton-Li)

Let V be a subspace of c_0 with (MAP). Then for every $\varepsilon > 0$, there exist $(E_n)_n$ a sequence of finite dimensional subspaces of V^* and a ω^* to ω^* continuous linear map $T : V^* \to (\sum \oplus_n E_n)_{\ell_1}$ such that : $\forall x^* \in V^* : (1-\varepsilon)\|x^*\| \leq \|Tx^*\| \leq (1+\varepsilon)\|x^*\|$.

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Theorem (Grothendieck)

Let X be a Banach space. Then :

i) *If X^* has (MAP) then X has (MAP).*

ii) *If X^* is separable and has (AP) then X^* has (MAP).*

Questions

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ii) $S_0(M)$ is 1-norming + M proper $\Longrightarrow \mathcal{F}(M)$ has (AP) ?

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$X \widehat{\otimes}_\pi Y = \overline{\text{span}}^{\|\cdot\|}\{x \otimes y \,:\, x \in X,\, y \in Y\} \subseteq \mathcal{B}(X \times Y, Z)^*$.

Proposition (Fundamental linearisation property)

Consider B in $\mathcal{B}(X \times Y, Z)$. Then there exists a unique continuous linear operator $\bar{B} : X \widehat{\otimes}_\pi Y \to Z$ such that $\|\bar{B}\| = \|B\|$ and such that the following diagram commutes

$$\begin{tikzcd}
X \times Y \arrow[r, "B"] \arrow[d, hook] & Z \\
X \widehat{\otimes}_\pi Y \arrow[ur, "\overline{B}"'] &
\end{tikzcd}$$

Thus $\mathcal{B}(X \times Y, Z) \equiv \mathcal{L}(X \widehat{\otimes}_\pi Y, Z)$.

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i) $Z = \mathbb{R} : (X\widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$.

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This leads us to the following definition.

Definition (Vector-valued Lipschitz-free space)

We may define the X-valued Lipschitz-free space over M to be : $\mathcal{F}(M) \widehat{\otimes}_\pi X$.

Injective tensor product : We have chosen to define $x \otimes y$ as an element of $\mathcal{B}(X \times Y)^*$. But we can use another point of view.

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For $(x^*, y^*) \in X^* \times Y^*$ define $x \otimes y$ by :
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Now let :

$$X \widehat{\otimes}_\varepsilon Y = \overline{\mathrm{span}}^{\|\cdot\|} \{x \otimes y \; : \; x \in X, \, y \in Y\} \subseteq \mathcal{B}(X^* \times Y^*).$$

In some cases, we have the relation $S_0(M)^* \equiv \mathcal{F}(M)$ and thus $Lip_0(M) \equiv S_0(M)^{**}$. What about the vector-valued case ?

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Proposition (Tensor product theory)

i) *X^* or Y^* has (RNP) and X^* or Y^* has (AP)* $\Longrightarrow$ $(X \widehat{\otimes}_\varepsilon Y)^* \equiv X^* \widehat{\otimes}_\pi Y^*$.

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Theorem (García-Lirola, Rueda Zoca, P.)

If M is a proper metric space, then $S_0(M,X) \equiv \mathcal{K}_{\omega^*,\omega}(X^*, S_0(M))$. Thus if $S_0(M)^* \equiv \mathcal{F}(M)^*$, and if $\mathcal{F}(M)$ or X^* has (AP) then $S_0(M,X)^* \equiv \mathcal{F}(M)\widehat{\otimes}_\pi X^*$ and $S_0(M,X)^{**} \equiv Lip_0(M,X)$.

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Study of $\mathcal{F}(M)\widehat{\otimes}_\pi X$? Two points of view : Tensor product theory and Lip functions theory (depending on the property studied).

1. Notation
2. Lipschitz-free spaces
   - Definition and first properties
   - Little Lipschitz spaces and double duality results
   - Around some ℓ_1 properties
3. Vector valued case
   - Projective tensor product
   - Injective tensor product and bi-duality results
   - Natural questions
   - Norm attainment

The identification $f \in Lip_0(M, X) \to \bar{f} \in \mathcal{L}(\mathcal{F}(M), X)$ raise many natural questions. For instance :

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In $\mathcal{L}(\mathcal{F}(M),X)$ we have a clear notion of norm attainment for $\bar{f}$: $\exists \gamma \in \mathcal{F}(M)$ such that $\|\bar{f}(\gamma)\|_X = \|\bar{f}\|$ ($\rightarrow \mathcal{NA}(\mathcal{F}(M),X)$).

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Definition (Strong norm attainment)

We say that $f \in Lip_0(M,X)$ strongly attains its norm if there exists $x \neq y$ such that $\|f(x) - f(y)\|_X = \|f\|_{Lip} d(x,y)$. We denote $Lip_{SNA}(M,X)$ the set of all Lipschitz functions which strongly attain their norm.

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Theorem (García-Lirola, Rueda Zoca, P.)

i) *Let* M *be a proper m. s. such that* $S_0(M)^* \equiv \mathcal{F}(M)$*. Then* $\mathcal{NA}(\mathcal{F}(M), X) = Lip_{SNA}(M, X)$

ii) *Let* M *be a proper m. s. such that* $S_0(M)^* \equiv \mathcal{F}(M)$*. Assume that* $\mathcal{F}(M)$, X^* *have (RNP), and* $\mathcal{F}(M)$ *or* X^* *has (AP), then :* $\overline{\mathcal{NA}(\mathcal{F}(M), X^{**})}^{\|\cdot\|} = \mathcal{L}(\mathcal{F}(M), X^{**})$ *and* $\overline{Lip_{SNA}(M, X)}^{\|\cdot\|} = Lip_0(M, X^{**})$.

Thank you very much !

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https://arxiv.org/abs/1603.01391

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