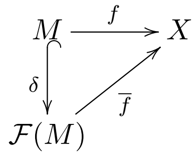
AAPG2020	FRII (FRee space Isomorphisms and Isometries)		JCJC
Coordinated by:	Antonín PROCHÁZKA	4 years	99 900 €
CES40			

1. Pre-proposal's context, positioning and objectives

This project aims at substantially advancing the knowledge about Lipschitz-free spaces and their applications to metric geometry and to functional analysis. For a metric space (M,d) the Lipschitz free space (here, for brevity, just free space) $\mathcal{F}(M)$ is a Banach space that is built around the metric space M in such a way that M is isometric to a subset $\delta(M)$ of $\mathcal{F}(M)$, and Lipschitz maps from M into any other Banach space X canonically become bounded linear operators from $\mathcal{F}(M)$ into X. In particular $\mathcal{F}(M)$ is a canonical isometric predual of the space $\mathrm{Lip}_0(M)$ of Lipschitz functions on M. But also, given two metric spaces M and N, Lipschitz maps from M into N become bounded linear operators from $\mathcal{F}(M)$ into $\mathcal{F}(N)$.



Naturally, the study of free spaces is at the intersection of functional analysis and metric geometry. In this project we are particularly interested in the interplay of these two domains. Nevertheless, free spaces are studied by several groups of researchers, for different reasons and under different names. Thanks to the Kantorovich-Rubinstein duality theorem, the norm on $\mathcal{F}(M)$ can be interpreted as the cost of the optimal solution of a certain transportation problem. Therefore the free spaces are used in disguise in PDE's, computer vision, image retrieval, etc. They are also of significant interest for computer science where the names that are commonly used for this distance are earth mover distance and transportation cost. For example, in [22] the authors used deep Banach space theoretic techniques to show that $\mathcal{F}(\mathbb{R}^2)$ does not embed into $\mathcal{F}(\mathbb{R}) \equiv L^1$ and therefore proved lower bounds for running times of certain type of algorithms related to similarity of 2D-images and nearest neighbor search. The natural question whether $\mathcal{F}(\mathbb{R}^3)$ embeds into $\mathcal{F}(\mathbb{R}^2)$ has resisted all attempts of solution despite the fact that the answer would be interesting both from theoretical and applied viewpoint.

The free spaces have been (re)introduced and revisited by different authors during the second half of the 20th century (Arens–Eells, Kantorovich-Rubinstein, Michael, J.A. Johnson, Pestov, Kadets...) who (re)discovered some of their basic features. A systematic study of free spaces themselves (under the name Arens–Eells spaces) started in the work of Weaver (publication of the authoritative monograph [28] in 1999). However the exponentially growing activity in the domain observed nowadays really took off after the publication of the ground-breaking paper of Godefroy and Kalton [19] where spectacular applications of free spaces to non-linear geometry of Banach spaces were exhibited in this paper and clear incentives for the study of the linear structure of free spaces were made. For instance, showing that $\mathcal{F}(\ell_1)$ is complemented in its bidual would settle the long-standing open problem of unique Lipschitz structure of ℓ_1.

Some successful examples bridging between functional analysis and metric geometry follow. Godard has proved a characterization of free spaces over subsets of $\mathbb{R}$-trees. Building on his work, we characterized the metric spaces M such that isometrically $\mathcal{F}(M) = \ell_1$ [13], resp. (almost-)isometrically $\mathcal{F}(M) \subseteq \ell_1$ [3]. Improving on a partial result due to Ivakhno, Kadets and Werner, we have fully characterized free spaces with Daugavet property as those coming from length spaces [16]. Full characterization of octahedral free spaces is accomplished in [24] introducing a brand new geometric property of metric spaces. It turns out that all infinite subsets of ℓ_1 have this property, which is not the case for ℓ_p, $p \neq 1$, and for infinite-dimensional CAT(κ) spaces. Unified theory of preduals [17] allows for constructions of new examples of free spaces which are duals as well as for applications to extremal structure of the unit ball of $\mathcal{F}(M)$. The full metric description of strongly exposed points of the unit ball of any $\mathcal{F}(M)$ in [16] and applications thereof to norm-attainment of linear operators in $\mathcal{L}(\mathcal{F}(M), X)$ inspired further intensive research in metric geometry [5] resp. in Bishop-Phelps-Bollobas properties [8, 9, 10]. The partial results in [17, 3, 4] seem to indicate that full description of extreme points of the unit ball of $\mathcal{F}(M)$ is behind the corner.

Examples of Banach spaces enjoying curious geometric properties can be often constructed easily using free spaces [21, 25, 6, 14]. We would like to capitalize this observation in a pioneering study of linear dynamics in free spaces.

Nevertheless, many basic questions about free spaces stay painfully without answer. We believe that the time is ripe for this to change since several new ideas appeared recently (supports [2, 4], quotient description of $\mathcal{F}(\mathbb{R}^n)$ [12]) with the potential to give the needed push.

Objective 1 *Study the weak convergence and weakly compact sets in free spaces.*

Many of the problems remain unsolved because of the poor understanding of weak convergence and of weakly compact sets in free spaces in general. The newly introduced notion of support should help to unblock the situation. A progress in understanding the weak convergence will help to characterize for which M the free space $\mathcal{F}(M)$ enjoys the Schur property (sufficient conditions given in [23] but are not necessary (unpublished)). This in turn should help in deciding whether in free spaces the Schur property and the Radon-Nikodým property always coincide (recently proved for free spaces over subsets of $\mathbb{R}$-trees [3]). Further problems whose solution depend on the description of weak compacts appear in the next objective.

Objective 2 *Which Banach space properties of L^1 stay valid for $\mathcal{F}(\mathbb{R}^n)$? Which Banach space properties of L^∞ stay valid for $Lip_0(\mathbb{R}^n)$?*

This question is motivated by the recent discoveries that $\mathcal{F}(\mathbb{R}^n)$ are weakly sequentially complete (WSC) [11] and complemented in their respective biduals [12, 20] (properties known for L^1) and that $\mathrm{Lip}_0(\mathbb{R}^n) \simeq \mathrm{Lip}_0(\mathbb{Z}^n)$ [7] (analogue of the fact that $L^\infty \simeq \ell_\infty$). Let us mention also that the points of Gâteaux differentiability of the norm on $\mathrm{Lip}_0(M)$ are exactly the points of Fréchet differentiability of the norm [16] (well known property of ℓ_∞ and L^∞). The other properties of L^1 which we have in mind are: **a)** cotype 2 (Whether $\mathcal{F}(\mathbb{R}^n)$ has a nontrivial cotype is a long-standing open problem. It has its place on this list but it is perhaps the hardest one. **b)** Strongly Weak Compact Generation (This is a stronger condition than (WSC). It is imperative to understand the weakly compact sets of $\mathcal{F}(\mathbb{R}^n)$ cf. Objective 1.) **c)** Dunford-Pettis Property (Again, the description of weakly compact sets is necessary preliminary step.) **d)** L-embedded (This isometric property would imply Property (X) which again is stronger than (WSC) (whether $\mathcal{F}(\mathbb{R}^n)$ enjoys (X) is asked in [20]). Notice that it is proved in [24] that $\mathcal{F}(\ell_p)$ is not L-embedded when $p > 1$.)

Objective 3 *Does there exist an infinite dimensional Banach space X such that $\mathcal{F}(X)$ embeds linearly into $\mathcal{F}(\mathbb{R}^n)$?*

This question is closely linked to the previous one. Even though it has not been formulated in print, this is a question that many experts are aware of. It is not even known whether X could be taken among the classical sequence spaces ℓ_p. Of course, the first instinct is to try and solve the question in the negative. For this the properties mentioned in Objective 2 could be useful. That the instincts can be deceitful is shown in [18] where an example is given of a compact set K and an infinite dimensional Banach space such that $\mathcal{F}(K) \simeq \mathcal{F}(X)$. Moreover it is not known the $\mathcal{F}(K) \simeq \mathcal{F}(X)$ situation can happen for every Banach space X! A positive solution would imply that $\mathcal{F}(\ell_p)$ is WSC for $1 < p < \infty$.

Objective 4 *Characterize those separable metric spaces M such that $\mathcal{F}(M)$ is isometric to a dual space.*

Partial results in this direction have been obtained already in [17]. Sufficient conditions found in the work of Weaver, Kalton and Dalet are put into a common perspective and generalized. At the same time a non-trivial example of M is found in [17] such that $\mathcal{F}(M)$ is not isometrically a dual space. (The example is actually quite simple, the non-triviality here refers to the fact that $L^1 \not\subseteq \mathcal{F}(M)$. Indeed, up until then the condition $L^1 \subseteq \mathcal{F}(M)$ was the only known obstacle for non-duality of $\mathcal{F}(M)$.) Another example of, for fundamentally different reason, non dual $\mathcal{F}(M)$ appears in [6]. Answering this question in the context of uniformly discrete spaces would be already very interesting.

Objective 5 *Answer in general the question whether the extreme (or exposed) points of $B_{\mathcal{F}(M)}$ are contained in the set V of molecules, i.e. the elements of $\mathcal{F}(M)$ of the form $\frac{\delta(x)-\delta(y)}{d(x,y)}$, $x \neq y \in M$.*

Some partial progress on this question was made in [17] where we have shown that if $\mathcal{F}(M)$ is isometrically a dual space and satisfies some additional requirements, then the answer is affirmative. See also [3] for an affirmative answer for subsets of $\mathbb{R}$-trees. We plan to study this question in the wider context of extreme points of norming subsets of Banach spaces, opening the doors to new results in areas such as generalized probability theories or tensor products of cones.

The study of extreme points has immediate applications in norm-attainment of operators (see [17, 8, 9, 10]) and thus in optimization. For example, in [27] many existing algorithms used in artificial intelligence are represented as a separating hyperplane problem in an appropriate free space. In this paper the authors also ask for efficient ways to compute the norms of given Lipschitz functions. Since the functions which attain their norm attain it necessarily on some extreme point, it is clear that solving this objective in the affirmative would improve the efficiency of this computation.

Objective 6 *Study hypercyclic (and related) operators on free spaces.*

As we already mentioned, number of curious examples in Banach space theory can be constructed (often in simpler way) using the free spaces. Moment's reflections shows that this is also the case for the Rolewicz's hypercyclic operator on ℓ_1. More precisely there exists M such that $\mathcal{F}(M) \equiv \ell_1$ and there is Lipschitz map $f : M \to M$ such that $\overline{f} : \mathcal{F}(M) \to \mathcal{F}(M)$ is hypercyclic. There also exists a M such that $\overline{f}$ is not hypercyclic for any Lipschitz self-map of M. It seems within the reach to quickly come up with the characterization of such metric spaces. There is a big variety of dynamical notions that can be studied following the same program: from the weaker cyclicity to the much stronger frequent hypercyclicity where Ergodic Theory comes into play. For example, the construction of first supercyclic operators ([15, 26]) is notorious for its combinatorial difficulty. We would like to explore whether some of the heavy combinatorial machinery could be out-sourced into a relatively simpler behaviour of a Lipschitz self-map of a suitably chosen M. To the best of our knowledge, there is no work in this direction so far.

2. Presentation of the team, budget and organization of the project

Antonín Procházka (P.I.), Assistant professor (MCF) at Université de Franche-Comté, since 2010. http://lmb.univ-fcomte.fr/Prochazka. Involvement 100%. A.P. defended his PhD in 2009 in Prague. **Experience as coordinator of projects:** coordinator of the bilateral project Chili-France ECOS C14E06 from 2015 till 2017. Co-organizer of several conferences. **Experience in the scientific field** 8 research articles related to free spaces, 4 in non-linear geometry, co-direction of thesis of Colin Petitjean dedicated to free spaces. **Project's capacity to promote the coordinator's level of responsibility:** A.P. is going to be responsible for recruiting and mentoring a post-doc as well as for organization of a conference in the year 3 of the project. A.P. is going to prepare the habilitation (HDR) during the duration of the project.

Colin Petitjean, Assistant professor (MCF) at Université Paris-Est Marne-la-Vallée, since 2019. http://cpetit13.perso.math.cnrs.fr. Involvement 100%. 6 research articles related to free spaces, 2 more in non-linear geometry, 1 in optimization.

Romuald Ernst, Assistant professor (MCF) at Université du Littoral Côte d'Opale, since 2016. http://ernst.r.perso.math.cnrs.fr. Involvement 25% (R.E. is already involved in ANR-17-CE40-0021 "Front"). 9 research articles in linear dynamics.

Budget:

	Details	**Cost**
Missions	1,5k€/year A.P., 1,5k€/year C.P., 500€/year R.E.	14k€
Invitations	2,2k€/year A.P. 1,8k€/year C.P., 500€/year R.E.	18k€
Post-doc	12 months in year 2 (includes her/his missions)	48k€
Conference	1 conference during the year 3 of the project	10k€
Material	Books, laptops, ...	2,5k€
Total	includes the administrative cost (+8%)	99,9k€

Organization of the project We will start or continue a collaboration with other young mathematicians. This applies in particular but is not limited to L.C. García-Lirola (Kent), A. Rueda Zoca (Granada), E. Pernecká and M. Cúth (Prague), R. Aliaga (Valencia), P. Kaufmann (São Paulo). Objectives 1 to 5 can be readily attacked by C.P. and A.P. Objective 6 will require the expertise of R.E. in linear dynamics as well as that of C.P. and A.P. in free spaces.

Recruitment of a **post-doc** in year 2 of the project. It is to be expected that the results obtained during the 1st year will offer, on one hand, many opportunities to be applied in wide range of situations and, on the other hand, many opportunities for further fine-tuning and generalizations. This will constitute a basis for the future post-doc's research.

Organization of an **international conference** in year 3 of the project. Up to our knowledge this will be the first ever conference dedicated exclusively to free spaces. We will pay special attention to bringing together researchers from complementary fields (optimal transport, metric geometry, functional analysis, computer science) who do not usually meet at their thematic conferences. As an aftermath of the conference, new collaborations are to be expected in year 4 of the project.

Special effort will be made in order to achieve equality of chances in the recruitment process and parity in the selection of speakers for the conference.

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